6. Reduce the heliocentric argument of the latitude, $u=\omega+\eta$ (i.e. the longitude of the body measured from the ascending node) to the ecliptic plane.

Solution: Since l is the heliocentric longitude measured from the vernal equinox, $l-\Omega$ in the ecliptic plane corresponds to u in the orbital plane. With i being the inclination between orbit and ecliptic, it is $\tan u=\tan(l-\Omega)/\cos i$. If one has the relation $\tan x=a\tan y$ where $a\to 1$, it follows:

$$\tan(x-y) = (a-1)\tan y/(1+a\tan^2 y)$$
$$= 2(a-1)\sin y\cos y/(2\cos^2 y+2a\sin^2 y)$$

or

$$\tan(x-y) = [(a-1)/(a+1)]\sin 2y/(1-[(a-1)/(a+1)]\cos 2y)$$

and, therefore,

$$x-y = \frac{a-1}{a+1}\sin 2y+\tfrac{1}{2}\left(\frac{a-1}{a+1}\right)^2\sin 4y+\cdots$$

or

$$u-(l-\Omega) = \frac{\tan^2\frac{1}{2}i}{\sin 1''}\sin 2(l-\Omega)$$

The maximum difference between u and $l-\Omega$ is obtained for $l-\Omega=45°$ with $\tan^2\frac{1}{2}i/\sin 1''$. For most planet orbits the maximum difference is very small; e.g. for Mars $\tan^2\frac{1}{2}i/\sin 1''=52''.8$. Astronautically, however, this difference is not negligible. Since $1''=410$ n. mi./A.U., $52''.8$ at a mean Martian distance of 1.52 A.U. corresponds to 32,800 n. mi. For Mars $\Omega\sim 49°$, hence $l=94°$ for the maximum difference in longitude.

7. Discuss the effect of the orbital inclination of Mars on the planet's deviation from the ecliptic plane and determine the approximate date of departure from Earth at which the miss distance from Mars in heliocentric latitude due to a $\Delta\eta=180°$ transfer would be a maximum (assume circular planet orbits at mean heliocentric distances).

Answer: The maximum elevation of Mars above the ecliptic plane is equal to $i\approx 1°51'=6{,}600''$ or $4.15\cdot 10^6$ n. mi., at $u=90°$ or $l\approx 139°$. In the case of a $\Delta\eta=180°$ transfer ellipse to Mars the terrestrial position at departure would then be $l=139-180°=-41°=319°$. At a terrestrial angular velocity of about 1°/d, the departure time would thus be about 41 days prior to the Earth's position at the vernal equinox, hence, the Sun's position at the autumnal equinox, or ~ 41 days prior to Sept. 23, or about August 13.

CHAPTER 6

PERTURBATIONS

6-1 INTRODUCTION

The theory of motion of a body in a non-central force field is called *perturbation theory*. The deviations from a central force field, which may vary in time and space, are called *perturbations*. A treatment of the mathematically involved and complex perturbation theory is felt to be mandatory in a book of this type, in recognition of the fact that there is really no such thing as an undisturbed flight path. In the Earth-Moon field the perturbations are in fact quite strong. The same is true in terrestrial space because of the oblateness of the Earth; in cislunar space because of the Moon, and, to a certain extent, the Sun; and in lunar space because of the Earth's gravity field. Travel within the solar system involves severe perturbations of the heliocentric transfer orbits by planetary fields at departure and arrival. In fact, rarely in astronomy do celestial bodies get so close together or become exposed, in certain parts of their flight paths, to such strong perturbative forces as in astronautics. Perturbation theory therefore will be as indispensable to the astronautical engineer and scientist as boundary layer theory is to the aerodynamicist.

Analogous to the conditions in aerodynamics it is permissible in space flight mechanics to neglect perturbations in some instances, but in the great majority of cases their effect has to be taken into account. If the perturbations due to oblateness are strong, as in the case of Earth satellites or a vehicle circumnavigating the Moon, it is impossible to describe the orbit without considering them. If the vehicle comes close to the Moon, even the performance requirements will be affected if subsequent return into a terrestrial satellite orbit is required. This is because of the correction maneuvers which must be carried out after leaving the space of strong lunar attraction. In interplanetary flights the conditions are less extreme because of the great distances involved between the planets and the overwhelming mass of the Sun. Central force field theory can specify rather closely the performance requirements and describe the transfer orbit. However, navigational requirements involve much higher accuracies in terms of position measurements to determine flight path errors or verify expected perturbations upon departure from or approach to a planet.

Astronavigation is impossible without accounting for perturbations. Unfortunately, like boundary layer theory or shifting equilibrium effects in the expansion of a dissociated gas in rocket nozzles, the perturbation effects require considerably more laborious computations than does the simplified analysis.

There is no "easy" way of mastering the perturbation theory. However, in cognizance of the over-all subject matter of this book, the attempt herein is to familiarize the reader only with the principal methods and the over-all procedures of perturbation analysis, omitting many modifications or adaptations of methods to special conditions, as well as many mathematical and computational details. The fact that some 15 paragraphs had to be devoted to perturbation theory illustrates the extent of subject material developed during the last two hundred years by many generations of scientists, among them some of the most brilliant and ingenious minds of history. In contrast to boundary layer theory and gas-kinetic equilibrium processes, the mechanics of perturbations are simple and completely understood, based solely on Newton's gravitational law. Therefore, the advent of electronic computers removes the last onus attached to perturbation analysis, namely, overwhelming computational labor.

Mathematical preparations for the subsequent perturbation analysis are provided in Par. 6-2 with regard to Lagrange's and Hamilton's formulation of the equations of motion, and in the Appendix of this chapter with respect to the Gauss-Encke Method of stepwise integration by the method of mechanical quadratures. The latter method has been placed in the Appendix because its lengthy and involved mechanics would unduly distract the reader from the basic line of thought.

If the system consists of three or more finite bodies, generally of $n > 2$ bodies (n-body problem), it can be shown that the center of mass of this system is at rest (or moves through space at constant speed with the system). However, the orbits of the individual bodies with respect to the center of mass are no longer conic paths. The deviations from the conic path are called *perturbations*.

Perturbations are classified as *periodic* and *secular perturbations*. The periodic perturbations are sometimes subdivided further into *short-period* and *long-period perturbations*.

Periodic perturbations are the result of periodically returning constellations of the bodies in their orbits. They affect the instantaneous position of the perturbed body. Their period is tied to the periods of revolution of the bodies involved. If a perturbation of this type is itself subject to a cyclic repetition, i.e., to a period of higher order, one speaks of a *long-period perturbation*. The classical example in the solar system is presented by the two largest planets, Jupiter and Saturn. Jupiter moves more than twice as fast in its orbit than Saturn. Each time Jupiter passes Saturn

(conjunction), the perturbation is particularly large. The conjunction occurs a little later every time, because Saturn moves on while Jupiter completes another revolution. Thus, the conjunction itself will ultimately complete one revolution. The period of the conjunction (about 883 years) represents the period of this long-period perturbation.

Perturbations, causing variations which are proportional to the time, are called *secular perturbations*. They can be regarded as cumulative changes produced by the periodic perturbations. The secular perturbations affect the shape and orientation of the orbit in space rather than the position of the body. There is no clear line separating long-period and secular perturbations. In many cases, the periods involved are so long that for comparatively small time intervals the long-period perturbation can be treated like a secular perturbation, simply growing with time. The variation of the eccentricity of the Earth's orbit is such a case. Presently the eccentricity is decreasing and will continue to do so for thousands of years to come. Eventually, however, the eccentricity will increase again.

Looking at the perturbations in the light of space flight theory, two aspects must be considered: Perturbations which affect the celestial bodies between which space flight is likely to occur; and perturbations of the space vehicles proper. In the first group secular perturbations are of no significance, except over hundreds or thousands of years of space flight. Periodic perturbations are not negligible and must be taken into account individually for the journeys considered.

With respect to space vehicles, secular as well as periodic perturbations are of importance. Any perturbation caused by gravitational forces is due to a distortion of the ideal central force field. This distortion can be caused not only by other celestial bodies, but also by unsymmetrical shape (oblateness) and/or inhomogeneous composition of the central body. These latter effects are particularly noticeable in the close vicinity of the central body where artificial satellites are more likely to occur than natural ones. Perturbation due to the oblateness of the central body is of the secular type in the sense that it changes the satellite orbit as such and in the sense that the variation is proportional with time. Periodic perturbations by other celestial bodies play a very small role for artificial satellites which are very close to the Earth. If the orbits are farther out, periodic perturbations caused by Sun and Moon have to be considered. In this case also, long-periodic perturbations occur as for the Moon. These perturbations, however, may not become significant within the technical period of operation of the satellite.

Aside from satellite motion, the *transfer motion* of space vehicles plays an important role in space flight. In this motion the vehicle follows an orbit which connects the departure body with the target body. Insofar as these paths involve a free, non-powered motion of the vehicle, they

represent a portion of a Keplerian conic section. The transfer times are less than one period in the respective orbit; i.e., very small time periods are involved by astronomical standards. The spacecraft may pass a celestial body more or less closely. Such a single pass, or *hyperbolic encounter* represents a *non-periodic* or *special perturbation*. The perturbative force is strong only for a short period, but the process is of a singular character and will not be repeated. As an example of this type of perturbation, one may consider a space vehicle leaving the Earth, bound for another planet and passing the Moon at very close distance. The lunar attraction would completely alter the vehicle's orbit of departure.

There exist two principal methods of analyzing perturbations. They are called the method of *general* (or absolute) *perturbations* and the method of *special perturbations*. In the general perturbation analysis either closed analytical expressions are developed from which the perturbation can be computed simply by inserting the time or a function thereof, or, more frequently, it is necessary to use the basic central force field solution and account for the perturbative force by means of infinite series which show a sufficiently rapid convergence over longer periods of time. The general method is more fundamental in its character and is of particular significance for exploring the mechanics of the solar system as a whole. However, it is usually too complicated and farfetched if astronomically less fundamental information is required (as will mostly be the case in space flight mechanics). Therefore, only the second method, that of special perturbations, will be treated in this book. The method of special perturbations simply describes the motion of the disturbed body from point to point in given time intervals by means of numerical analysis (stepwise integration of the equations of motion of a body subjected to a number of force fields). The starting point is a moment or date (epoch) at which the instantaneous position and motion are known and can be expressed in terms of an undisturbed conic for a short time before and after this date (osculation epoch). Stepwise integration is always a cumbersome method and lacks the mathematical elegance and general validity of the general perturbation analysis, since its results are valid only for the period under computation (unless sufficiently short periodicity permits one to follow a complete cycle numerically, in which case the result is also of general validity).[1] The special perturbation method does have the tremendous practical advantage of being immediately applicable to any flight path, any number and constellation of disturbing bodies and allowing one to attain any desired degree of accuracy. In space flight mechanics the special perturbation method is, therefore, indispensable. The different methods available for the analytical treatment of special perturbations are surveyed in Par. 6-7.

[1] So-called "periodic solutions"; cf. Par. 6-5.

6-2 LAGRANGE'S AND HAMILTON'S EQUATIONS OF MOTION

Before proceeding to the treatment of the three-body problem and the theory of perturbations, it appears desirable to present briefly the classical methods of Lagrange and Hamilton in deriving the equations of motion of a particle. These methods play an important role in higher celestial mechanics and, therefore, are bound to occur more frequently in the astronautical literature of the future as the theoretical studies advance into more details. Within the framework of this book, the enormous variety of analytical methods based on Lagrange, Hamilton, Jacobi, variation of constants, and so forth cannot be treated in any degree of completeness. For a thorough study, reference is made to textbooks on advanced analytical mechanics. However, the non-specialized reader may welcome a brief introduction to the principles of Lagrange's and Hamilton's analysis of particle dynamics.

(a) Lagrange's Equations of Motion. Consider, generally, a system of n bodies with the masses $m_1, m_2, \ldots, m_{n-1}, m_n$. The motion of these bodies, related to an absolute (space-fixed) coordinate system, is then determined by 3 n equations,

$$\left.\begin{aligned} F_{xi} &= \frac{\partial U}{\partial x_i} = m_i \ddot{x}_i \\ F_{yi} &= \frac{\partial U}{\partial y_i} = m_i \ddot{y}_i \\ F_{zi} &= \frac{\partial U}{\partial z_i} = m_i \ddot{z}_i \end{aligned}\right\} \qquad (6\text{-}1)$$

$$(i = 1, 2, 3, \ldots, n)$$

where the correlation between force and potential in a conservative system[2] is given by the equations of definition (4-3a) and the preceding three equations.

If the motion is three-dimensional, the number of coordinates is $x_{1\ldots n}, y_{1\ldots n}, z_{1\ldots n}$, or 3 n. In the course of orbit determination, it is frequently advisable to replace the absolute coordinates x, y, z, by other coordinates or variables. These may or may not involve time as a variable. One example is the conversion from rectangular to polar coordinates. Another example, involving the introduction of time as a variable, is the Jacobian coordinate system which will be used later in computing the motion of a space vehicle in the Earth-Moon system. The significance of the Lagrangian equations of the second type for

[2] In a *conservative system* the potential is not a function of time (because no dissipative forces such as drag exist). U is the work done to establish the given system of n bodies from a state of infinite dispersion. Therefore, the potential energy of the system is negative (cf. Eq. 4-3a)).

conservative forces lies in the fact that they facilitate the establishment of the differential equations in the new variables, because the motion is expressed in terms of the kinetic and potential energies of the system.

Let the generalized coordinates be q_k $(k=1, 2, 3, \ldots, f)$. Assume that the coordinates q_i of the body m_i are independent of each other. Between the absolute and the generalized position coordinates exists, then, the functional relation $x_i=x_i(q_k t)$, $y_i=y_i(q_k t)$, $z_i=z_i(q_k t)$ and between the velocities

$$\left.\begin{aligned} \dot{x}_i &= \sum_{k=1}^{f} \frac{\partial x_i}{\partial q_k}\dot{q}_k+\frac{\partial x_i}{\partial t} \\ \dot{y}_i &= \sum_{k=1}^{f} \frac{\partial y_i}{\partial q_k}\dot{q}_k+\frac{\partial y_i}{\partial t} \\ \dot{z}_i &= \sum_{k=1}^{f} \frac{\partial z_i}{\partial q_k}\dot{q}_k+\frac{\partial z_i}{\partial z} \end{aligned}\right\} \tag{6-2}$$

Without going into the mathematical details here, it may be explained that the physical significance of the procedure leading to the formulation of Lagrange's equations is to introduce the generalized coordinates into the equations stating the principles of d'Alembert,

$$\sum_{i=1}^{n} (F_{xi}-m_i\ddot{x}_i)dx_i = 0 \tag{6-3}$$

etc., for y and z. This principle states that the impressed forces F_x, y, z and the inertial forces $m\ddot{x}$, $m\ddot{y}$, $m\ddot{z}$ are always in equilibrium. Although this principle is basically already contained in Newton's third law of motion (action = reaction), its significance lies in the fact that d'Alembert used it to reduce dynamic problems to static problems.

Introduction of the generalized coordinates into d'Alembert's principle leads to the Lagrange equations of the second type in a system of conservative forces:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k} = 0 \tag{6-4}$$

where L is the Lagrangian (function) or the kinetic potential

$$L = T - U \tag{6-5}$$

T represents the total kinetic energy of the system:

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i v_i^2 = \tfrac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2) \tag{6-6a}$$

It can be shown that $T=\dot{q}_k\partial L/\partial\dot{q}_k$. Since $\partial U/\partial\dot{q}_k=0$ (cf. below),

$$T = \tfrac{1}{2}\sum_{k=1}^{n} \dot{q}_k\frac{\partial T}{\partial \dot{q}_k} \tag{6-6b}$$

U is the force function; for one body

$$U_1 = k^2\frac{m}{r}$$

two bodies

$$U_2 = k^2\frac{m_1m_2}{r_{12}}$$

r_{12} = radial distance between the two bodies; three bodies

$$U_3 = k^2\left(\frac{m_1m_2}{r_{12}}+\frac{m_2m_3}{r_{23}}+\frac{m_1m_3}{r_{13}}\right)$$

Generally, n bodies

$$U = \sum_{i=1}^{n-1}\sum_{j=2}^{n} k^2\frac{m_im_j}{r_{ij}} \quad (j>i) \tag{6-7}$$

Since $U=-mgr$ represents the potential energy of the system,[3] $T+U=L+2U=h=$ const. is simply an expression of the fact that, in a conservative system, the sum of kinetic and potential energies is constant. Since the potential energy depends only on distance from the respective center of attraction and not on the velocity, it holds that $\partial U/\partial\dot{q}_k=0$, whence the Lagrangian equation may also be written in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}+\frac{\partial U}{\partial q_k} = 0 \tag{6-8}$$

The total differential of the kinetic energy is

$$\frac{dT}{dt} = \sum_{k=1}^{n}\frac{\partial T}{\partial q_k}\dot{q}_k+\sum_{k=1}^{n}\frac{\partial T}{\partial \dot{q}_k}\ddot{q}_k \tag{6-9}$$

Multiplying Eq. (6-8) by $\dot{q}_k$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\dot{q}_k-\frac{\partial T}{\partial q_k}\dot{q}_k = -\dot{q}_k\frac{\partial U}{\partial q_k} \tag{6-10}$$

forming the total differential of T as defined in Eq. (6-6b),

$$2\frac{dT}{dt} = \sum_{k=1}^{n}\frac{\partial T}{\partial \dot{q}_k}\ddot{q}_k+\sum_{k=1}^{n}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\dot{q}_k \tag{6-11}$$

and subtracting (6-9) from (6-11) yields the left side of Eq. (6-10). Now $(-\dot{q}_k\partial U/\partial q_k)$ represents the time rate at which the forces applied to the system do work, $-\dot{q}_k(\partial U/\partial q_k)\equiv dW/dt$. Therefore, the identity of the two equations means $dT/dt=dW/dt$, i.e., the change in kinetic energy of the system is equal to the time rate at which the applied forces do work.

[3] Here g is the gravitational acceleration at distance r, i.e., $g=g_r$. If $g=g_{00}$, i.e., the surface value, one must write $U=mg_{00}r_{00}^2/r$ for the potential at r so that the attractive force at r becomes $F_r=mg_r=mg_{00}r_{00}^2/r^2$.

In a conservative system, the force function is represented by $U(r)$ only, and, therefore, $dA/dt = -dU/dt$, whence

$$\frac{dT}{dt} = -\frac{dU}{dt} \tag{6-12}$$

Kinetic energy can be gained only at the expense of potential energy and vice versa. Their sum is constant in a conservative system. The equation of conservation of energy

$$T_1 + U_1 = T_2 + U_2 \tag{6-13}$$

is therefore an integral of Lagrange's equation.

(b) Application of Lagrange's Equations. *First Example*: Assume a free mass point $m(x, y, z)$. Then

$$q_1 = x, \quad q_2 = y, \quad q_3 = z$$
$$\dot{q}_1 = \dot{x}, \quad \dot{q}_2 = \dot{y}, \quad \dot{q}_3 = \dot{z}$$

The kinetic energy of m is then

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{6-14}$$

Furthermore, U is a function of x, y, z. Therefore,

$$-\frac{\partial U}{\partial x} = F_x, \quad -\frac{\partial U}{\partial y} = F_y, \quad -\frac{\partial U}{\partial z} = F_z$$

Consequently, for the x-coordinate

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial q_1} = -F_x, \quad \frac{\partial U}{\partial \dot{x}} = 0, \quad \frac{d}{dt}\left(\frac{\partial U}{\partial x}\right) = 0$$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}$$

The equation of motion in the x-coordinate becomes then, with the aid of Eq. (6-4),

$$\frac{d}{dt}\left(\frac{\partial(T-U)}{\partial \dot{x}}\right) - \frac{\partial(T-U)}{\partial x} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x}$$
$$= m\ddot{x} - 0 - 0 - F_x = 0$$

which, of course, is the x-component of the equation of motion $F_x = m\ddot{x}$. Obviously, the corresponding relations are obtained for the y- and z-coordinates. This result does not contain anything new, but rather illustrates the method of application of Lagrange's equation.

Second Example: Consider the motion of a planet in rectangular heliocentric ecliptic coordinates X, Y, Z, and in corresponding polar coordinates R, l, b. Then

$$\begin{aligned}
q_1 &= R, \quad q_2 = b, \quad q_3 = l \\
\dot{q}_1 &= \dot{R}, \quad \dot{q}_2 = \dot{b}, \quad \dot{q}_3 = \dot{l} \\
X &= R\cos b\cos l \\
Y &= R\cos b\sin l \\
Z &= R\sin b \\
\dot{X} &= \dot{R}\cos b\cos l - (R\sin b\cos l)\dot{b} - (R\cos b\sin l)\dot{l} \\
\dot{Y} &= \dot{R}\cos b\sin l - (R\sin b\sin l)\dot{b} + (R\cos b\cos l)\dot{l} \\
\dot{Z} &= \dot{R}\sin b + (R\cos b)\dot{b}
\end{aligned}$$

For the kinetic energy

$$T = \frac{m}{2}[\dot{R}^2 + R^2\dot{b}^2 + (R^2\cos^2 b)\dot{l}^2] \tag{6-15}$$

$$U = U(R, b, l)$$

Therewith, for the first coordinate

$$\frac{\partial U}{\partial q_1} = \frac{\partial U}{\partial R} = -F_R$$

$$\frac{\partial T}{\partial R} = mR\dot{b}^2 + m(R\cos^2 b)\dot{l}^2$$

$$\frac{\partial T}{\partial \dot{q}_1} = m\dot{R}; \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = m\ddot{R}$$

First equation of motion:

$$m\ddot{R} - mR\dot{b}^2 - mR\cos^2 b\dot{l}^2 + \frac{\partial U}{\partial R} = 0 \tag{6-16}$$

For the second coordinate

$$\frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial b}$$

$$\frac{\partial T}{\partial b} = (mR^2\sin b\cos b)\dot{l}^2$$

$$\frac{\partial T}{\partial \dot{b}} = mR^2\dot{b}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{b}}\right) = 2mR\dot{R}\dot{b} + mR^2\ddot{b}$$

Second equation of motion:

$$m[2R\dot{R}\dot{b}+R^2\ddot{b}-(R^2\sin b\cos b)\dot{l}^2] = 0 \tag{6-17}$$

For the third coordinate

$$\frac{\partial U}{\partial q_3} = \frac{\partial U}{\partial l}$$

$$\frac{\partial T}{\partial \dot{l}} = 0, \quad \frac{\partial T}{\partial l} = (mR^2\cos^2 b)\dot{l}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{l}}\right) = (2mR\dot{R}\cos^2 b)\dot{l}-(2mR^2\sin b\cos b)\dot{b}\dot{l}+(2mR^2\cos^2 b)\ddot{l}$$

Third equation of motion:

$$m[(2R\dot{R}\cos^2 b)\dot{l}-(2mR^2\sin b\cos b)\dot{b}\dot{l}+(2mR^2\cos^2 b)\ddot{l}] = 0 \tag{6-18}$$

Third Example: Plane motion in polar coordinates radius vector and true anomaly

$$x = r\cos\eta$$

$$y = r\sin\eta$$

$$T = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) = \tfrac{1}{2}m(\dot{r}^2+(r\dot{\eta})^2) \tag{6-19}$$

First coordinates:

$$\frac{\partial U}{\partial q_1} = -\frac{\partial U}{\partial r} = -F_r = -mg$$

$$\frac{\partial T}{\partial r} = mr\dot{\eta}^2, \quad \frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = m\ddot{r}$$

First equation of motion (radial force):

$$m(\ddot{r}-r\dot{\eta}^2) = F_r = mg \tag{6-20}$$

mg = gravitational force; $mr\dot{\eta}^2$ = apparent centrifugal force;
$m\ddot{r}$ = resultant radial force

Second coordinates:

$$\frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial \dot{\eta}} = rm\ddot{\eta} = rF_\eta$$

$$\frac{\partial T}{\partial \eta} = 0, \quad \frac{\partial T}{\partial \dot{\eta}} = mr^2\dot{\eta}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\eta}}\right) = mr^2\ddot{\eta}+2mr\dot{r}\dot{\eta}$$

Second equation of motion (azimuthal acceleration):

$$m(r^2\ddot{\eta}+2r\dot{r}\dot{\eta}) = rF_\eta \tag{6-21}$$

The left side is the time derivative of the angular momentum $mr^2\dot{\eta}$; the right side is the applied torque.

In the one-body or two-body case, the potential is, of course, a function of r only. Then $F_\eta = 0$, whence (6-21) becomes $d/dt(\partial T/\partial\dot{\eta}) = d/dt(mr^2\dot{\eta}) = 0$. The integral is $mr^2\dot{\eta} = p_\eta$, where p_η is the constant angular momentum (conservation of angular momentum). Since $r^2\dot{\eta}$ is twice the *areal velocity*, i.e., the area swept by the radius vector per unit time $\frac{1}{2}r(r\dot{\eta}) = dA/dt$, the conservation of angular momentum is therefore equivalent to stating that the areal velocity is constant. Conversely, the second Kepler law is an expression of the conservation of the angular momentum.

(c) Conjugate Momenta. The term $\partial T/\partial\dot{q}_k$ or, more generally, $\partial L/\partial\dot{q}_k$ is called *conjugate momentum*, *canonical momentum*, or simply *generalized momentum associated* with the coordinate q_k:

$$p_k = \frac{\partial L(q_k, \dot{q}_k, t)}{\partial \dot{q}_k} \tag{6-22}$$

Its derivative follows from Lagrange's equation,

$$\dot{p}_k = \frac{\partial L}{\partial q_k} \tag{6-23}$$

If q_k is not a cartesian coordinate, p_k does not necessarily have the dimensions of a linear momentum (mv). In the preceding third example, these conjugate momenta were

$$p_r = m\dot{r}, \quad p_\eta = mr^2\dot{\eta}$$

and in the second example,

$$p_R = m\dot{R}, \quad p_b = mR^2\dot{b}, \quad p = (mR^2\cos^2 b)\dot{l}$$

(d) Hamilton's Equations of Motion. The Hamiltonian equations of motion, like those of Lagrange, add nothing to the physical knowledge but represent formal methods of dealing effectively with the often confusing variety of physical processes. The Hamiltonian formulation furnishes an alternate and, in many respects, more powerful means of working with physical principles involved in the processes under consideration.

The new equations of motion are based on the Hamiltonian (function)

$$H = T+U \tag{6-24}$$

which is connected with the Lagrangian by the relation

$$L = 2T-H \tag{6-25}$$

The Hamiltonian represents the total energy of the system, if it is conservative, namely, the sum of kinetic and potential energies. It holds

$$H(p_k, q_k, t) = \sum_{k=1}^{n} \dot{q}_k p_k - L(q_k, \dot{q}_k, t) \tag{6-26}$$

The differential of the Hamiltonian is given by

$$dH = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k} dq_k + \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt \tag{6-27}$$

or,

$$dH = \sum_{k=1}^{n} \dot{q}_k dp_k + \sum_{k=1}^{n} p_k d\dot{q}_k - \sum_{k=1}^{n} \frac{\partial L}{d\dot{q}_k} d\dot{q}_k - \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} dq_i - \frac{\partial L}{\partial t} dt$$

From the definition of the conjugate momentum, Eq. (6-22), it follows

$$\sum p_k d\dot{q}_k = \sum \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k \tag{6-28}$$

Thus, using Eq. (6-28) and (6-23) the second Eq. (6-27) can be written

$$dH = \sum_{k=1}^{n} \dot{q}_k dp_k - \sum_{k=1}^{n} \dot{p}_k dq_k - \frac{\partial L}{\partial t} dt \tag{6-29}$$

With the aid of both Eq. (6-27) the following relations can be obtained which are known as *Hamilton's canonical equations*

$$\left.\begin{aligned} \dot{p}_k &= -\frac{\partial H}{\partial q_k} = \frac{\partial T}{\partial \dot{q}_k} \\ \dot{q}_k &= \frac{\partial H}{\partial p_k} = \frac{\partial T}{\partial p_k} \end{aligned}\right\} \tag{6-30}$$

Writing the total time derivative of H in the form immediately apparent from the first Eq. (6-27),

$$\frac{dH}{dt} = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k} \dot{q}_k + \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} \dot{p}_k + \frac{\partial H}{\partial t} \tag{6-31}$$

substituting $\dot{q}_k$ and $\dot{p}_k$ from Eq. (6-30), it follows that

$$\frac{dH}{dt} = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k} \frac{\partial H}{\partial p_k} - \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} \frac{\partial H}{\partial p_k} + \frac{\partial H}{\partial t} \tag{6-32}$$

or

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{6-33}$$

If n bodies are considered, the canonical equations provide a set of $2n$

first order differential equations which replace the Lagrange equations. They describe the motion of the system. In the Lagrange equations the coordinates q and the velocities $\dot{q}$ are used, while in Hamilton's equations the coordinates q and the momenta p are taken as variables. The velocity component $\dot{q}_k$ along a given coordinate q_k is equal to the partial derivative of the Hamiltonian Equation (6-24) or (6-25) with respect to the corresponding momentum p_k, with $\dot{p}_k$ being equal to minus the partial derivative of the Hamiltonian with respect to the coordinate q_k. The Hamiltonian is identical with the total energy in a conservative system. The time is in both cases an additional variable, usually the independent one. In a conservative system, the procedure of solving the problem of a system's motion is first to write the Lagrangian $L(q, \dot{q}, t)$ as shown in the preceding examples, then to compute the conjugate momenta from Eq. (6-22), and finally, to find the Hamiltonian H defined by Eq. (6-26).

If the Lagrangian of a system does not contain a given coordinate q_k but, on the other hand, may or may not contain the velocity component $\dot{q}_k$, then this coordinate is termed *cyclic* or *ignorable*. As implied, by the name, cyclic coordinates occur in circulating or rotating systems. The Lagrangian equations of motion which correspond to ignorable coordinates —which will be designated by a prime—are, remembering Eq. (6-4),

$$\frac{d}{dt} \frac{\partial L}{\partial q'_k} = 0 \tag{6-34}$$

If a coordinate is ignorable, i.e., if it satisfies the condition (6-34), then it must be

$$\frac{\partial T}{\partial q'_k} = \text{constant} \quad \text{or} \quad \frac{\partial T}{\partial q'_k} = 0 \tag{6-35}$$

Conversely, condition (6-35) is not a sufficient condition for a coordinate to be ignorable, since, unless $\partial U / \partial q_k = 0$, the Lagrangian is not independent of q_k, even though T is. Therefore, the coordinates x, y, z, in the first example of Sect. (b), are not ignorable, whereas b and l in the second example and η in the third example are ignorable.

Writing the Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}'_k}\right) = \frac{dp_k}{dt} = 0$$

the integral becomes

$$\frac{\partial L}{\partial \dot{q}'_k} = \text{constant} = p'_k \tag{6-36}$$

or

$$\frac{\partial T}{\partial \dot{q}'_k} = \text{constant} \tag{6-37}$$

This shows that the cyclic impulse coordinates p'_k are integration constants. This has been illustrated already in the third example, Sect. (b) of this paragraph.

(e) Application of Hamilton's Equations. In the cases of interest to celestial mechanics, the Hamiltonian H represents the total energy of the system. Then the Hamiltonian can be found directly without first writing the Lagrangian.

Take the motion in a central force field according to the second example, Sec. (b) of this paragraph. The procedure is as follows: H is given by Eq. (6-24), T is given by Eq. (6-15). Then

$$p_R = \frac{\partial T}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{R}} = m\dot{R}; \quad m^2\dot{R}^2 = p_R^2$$

$$p_b = \frac{\partial T}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{b}} = mR^2\dot{b}; \quad \dot{b} = \frac{p_b}{mR^2}; \quad R^2\dot{b}^2 = \frac{p_b^2}{m^2R^2}$$

$$p_l = \frac{\partial T}{\partial \dot{q}_3} = \frac{\partial T}{\partial \dot{l}} = m(R^2\cos^2 b)\dot{l}; \quad \dot{l} = \frac{p_l}{mR^2\cos^2 b}$$

$$(R^2\cos^2 b)\dot{l}^2 = \frac{p_l^2}{m^2R^2\cos^2 b}$$

$$U = -\frac{K}{r}$$

Thus the Hamiltonian H becomes

$$H = \frac{1}{2m}\left(p_R^2 + \frac{p_b^2}{R^2} + \frac{p_l^2}{R^2\cos^2 b}\right) - \frac{K}{R} \tag{6-38}$$

Hamilton's equations of motion are

$$\left.\begin{aligned}
\frac{\partial H}{\partial p_R} &= \dot{R} = \frac{p_R}{m} \\
\frac{\partial H}{\partial p_b} &= \dot{b} = \frac{p_b}{mR^2} \\
\frac{\partial H}{\partial p_l} &= \dot{l} = \frac{p_l}{mR^2\cos^2 b} \\
-\frac{\partial H}{\partial R} &= \dot{p}_R = \frac{p_b^2}{mR^3} + \frac{p_l^2}{mR^3\cos^2 b} - \frac{K}{R^2} \\
-\frac{\partial H}{\partial b} &= \dot{p}_b = -\frac{2p_l^2\sin b}{R^2\cos^3 b} \\
-\frac{\partial H}{\partial l} &= \dot{p}_l = 0
\end{aligned}\right\} \tag{6-39}$$

Now take the motion in a central force field according to the third example, Sec. (b) of the paragraph. In this case, one can easily show

$$\left.\begin{aligned}
H &= \frac{1}{2m}\left(p_r^2 + \frac{p_\eta^2}{r^2}\right) - \frac{K}{r} \\
\frac{\partial H}{\partial p_r} &= \dot{r} = \frac{p_r}{m}; \quad \frac{\partial H}{\partial p_\eta} = \dot{\eta} = \frac{p_\eta}{mr^2} \\
-\frac{\partial H}{\partial r} &= \dot{p}_r = \frac{p_\eta^2}{mr^3} - \frac{K}{r^2}; \quad -\frac{\partial H}{\partial \eta} = \dot{p}_\eta = 0
\end{aligned}\right\} \tag{6-40}$$

If the force function U depends only upon the coordinates and not on the time, the Hamiltonian represents the total energy of the system.

6-3 THE THREE-BODY PROBLEM

Consider three masses, m_1, m_2, m_3. The equations of motion of this system are

$$\left.\begin{aligned}
-k^2\sum_{i=1}^{2}\sum_{j=2}^{3}\frac{m_im_j}{r_{ij}} &= U = -k^2\left[\frac{m_1m_2}{r_{12}} + \frac{m_2m_3}{r_{23}} + \frac{m_1m_3}{r_{13}}\right](j>i) \\
m_i\ddot{q}_k &= -\frac{\partial U}{\partial q_k}
\end{aligned}\right\} \tag{6-41}$$

$ij = 12, 23, 13$, and $k = 1, 2, 3, \ldots 9$, corresponding to the nine cartesian coordinates x_1, y_1, z_1, and so forth. These are nine second order differential equations requiring 18 integrations. Putting

$$m_i\dot{q}_k = p_k \quad \begin{pmatrix} i = 1, k = 1, 2, 3 \\ i = 2, k = 4, 5, 6 \\ i = 3, k = 7, 8, 9 \end{pmatrix} \tag{6-42}$$

the Hamiltonian becomes

$$H = \tfrac{1}{2}\sum_{k=1}^{9}\frac{p_k^2}{m_i} + U \tag{6-43}$$

where U is given by the equation preceding Eq. (6-7). Bringing the equations into their canonical form according to Eq. (6-30) leads to 18 first-order differential equations. The system is, therefore, of the 18th order for the determination of the variables $p_1, p_2, \ldots, p_9$ and $q_1, q_2, \ldots, q_9$.

Let q_1, q_2, q_3, be the coordinates of the first body, q_4, q_5, q_6 of the second body, and q_7, q_8, q_9 of the third body, and identify the conjugate momenta in the same manner. The kinetic energy of the system is given by

$$T = \frac{m_1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + \frac{m_2}{2}(\dot{q}_4^2 + \dot{q}_5^2 + \dot{q}_6^2) + \frac{m_3}{2}(\dot{q}_7^2 + \dot{q}_8^2 + \dot{q}_9^2)$$

The integrals are (Eq. (6-41): $m_i\dot{q}_k = -\partial U/\partial\dot{q}_k + c = c$; $(\partial U/\partial\dot{q}_k = 0)$),

$$p_1+p_4+p_7 = c_1$$
$$p_2+p_5+p_8 = c_2$$
$$p_3+p_6+p_9 = c_3$$

and, upon integration of the velocities,

$$m_1q_1+m_2q_4+m_3q_7 = c_1t+C_1$$
$$m_1q_2+m_2q_5+m_3q_8 = c_2t+C_2$$
$$m_1q_3+m_2q_6+m_3q_9 = c_3t+C_3$$

These six equations state that the center of mass of the system is at rest or moves in a straight line at constant velocity through space. The physical reason for this is that no external forces act on the system; only the internal forces of mutual attraction exist and, according to Newton's third law, internal forces cannot change the motion of the center of mass of a system.

Since the theorem of conservation of angular momentum is valid in a system of bodies in motion under the influence of their mutual forces of attraction, it follows that the angular momentum of the system about each of the coordinate axes is also constant:

$$\frac{\partial T}{\partial\dot{q}_k} = \text{constant}$$

$$\frac{\partial T}{\partial\dot{q}_k} = \tfrac{1}{2}\frac{\partial}{\partial\dot{q}_k}\sum m_i(x_i^2+y_i^2+z_i^2)$$

$$\frac{\partial T}{\partial\dot{q}_k} = \sum m_i\left(\dot{x}_i\frac{\partial x_i}{\partial q_k}+\dot{y}_i\frac{\partial y_i}{\partial q_k}+\dot{z}_i\frac{\partial z_i}{\partial q_k}\right)$$

Writing $x_i = r_i\cos\theta_i$; $y_i = r_i\sin\theta_i$; one has $d\,q_k = d\theta_i$; and

$$\frac{\partial x_i}{\partial\theta_i} = -r_i\sin\theta_i = -y_i$$

$$\frac{\partial y_i}{\partial\theta_i} = r_i\cos\theta_i = x_i$$

$$\frac{\partial z_i}{\partial\theta_i} = 0$$

Therewith the above equations become

$$\frac{\partial T}{\partial\dot{q}_k} = \sum m_i(x_i\dot{y}_i - y_i\dot{x}_i)$$

or, since generally $m_i\dot{q}_k = p_k$ according to Eq. (6-42), the angular momentum of one body about one coordinate axis is $(q_{k-1}p_k - q_kp_{k-1})$ or, in the case of 3 bodies and 3 axes each,

$$q_1p_2-q_2p_1+q_4p_5-q_5p_4+q_7p_8-q_8p_7 = A_1$$
$$q_2p_3-q_3p_2+q_5p_6-q_6p_5+q_8p_9-q_9p_8 = A_2$$
$$q_3p_1-q_1p_3+q_6p_4-q_4p_6+q_9p_7-q_7p_9 = A_3$$

where A_1, A_2, A_3 are constants.

Finally, the tenth integral is obtained from the fact that U is a function of distance only, whence

$$H = T+U$$

$$\sum m_i(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2) = \sum m_iv_i^2 = 2U+h$$

which is the *vis viva* integral with h as integration constant (cf. Eq. (4-22) and (4-23)).

The 10 integrations so obtained are: 6 center of mass integrals, 3 area integrals (constant angular momentum), and the *vis viva* integral. The system is now of the 8th order. Lagrange has shown that the system can be reduced to the 6th order (or in the planar case ($z=0$) to the 4th order); but the last two reductions are not a consequence of integrations. Thus, only 10 algebraic integrals are known in the analysis of the general three-body problem. The last two reductions, down to the 6th order, are incident to special coordinate selection and elimination of time. These steps are not of great practical significance, because of the complexity of the resulting expressions.

Of greater importance for the present discussion is the fact that a complete and general integration of all equations of motion in the three-body problem is not known to be possible. It is therefore necessary to apply numerical methods such as expansion in series or numerical integration.

In two special cases periodic solutions of the three-body problem have been found without resorting to the simplification that one of the bodies possesses infinitesimal mass (cf. next paragraph). Both cases were found by Lagrange in 1772.

In the first case the 3 bodies describe orbits whose foci coincide with the barycenter, i.e., the center of mass (Lagrange's solution of the equidistant mass points). In the second case, the 3 masses lie in a straight line passing through the barycenter. The line rotates at a constant angular velocity about the barycenter (Lagrange's solution of the collinear mass points). Fig. 6-1 (a) and (b) are examples of the equidistant mass points; (c) for the collinear mass points. Those points for which Lagrange's solution is rigidly fulfilled are called *libration points*.

The equidistant mass point solution can be derived quickly from the requirement of the balance between gravitational and centrifugal forces

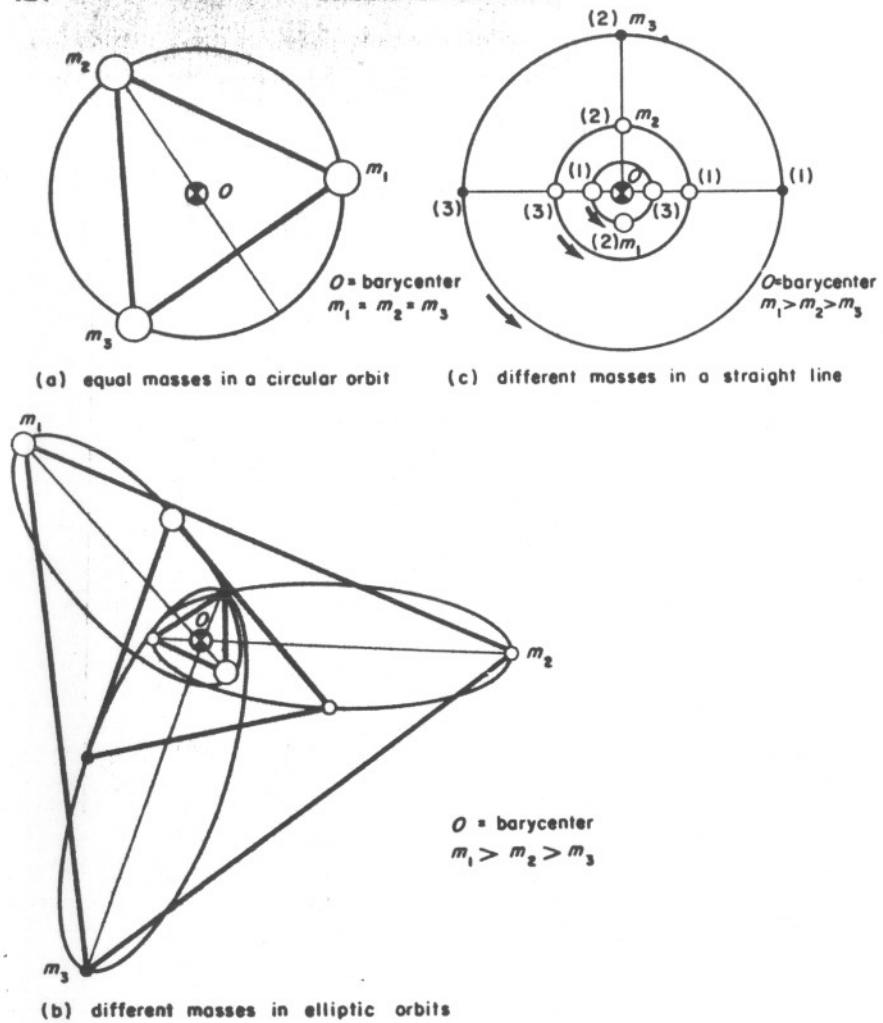

FIG. 6-1 Examples for Lagrange's Periodic Solutions of the Three-Body Problem.

acting upon the masses which rotate about the barycenter. Figure 6-2 shows the masses m_1, m_2, m_3 in an arbitrary triangle. The distance between the masses is r and the distance from the barycenter is s. The angles of the mass triangle are ψ; those between the radii vectors φ. The bodies revolve at the angular velocity μ about the barycenter. Then the requirement of equilibrium between attraction and centrifugal effect ($ms\mu^2$) at all three masses yields the three equations,

$$s_i^2\mu^4 = k^4\frac{m_j^2}{r_{ij}^4} + k^4\frac{m_k^2}{r_{ik}^4} + 2k^4\frac{m_jm_k}{r_{ij}^2r_{ik}^2}\cos\psi_i \tag{6-44}$$

where k^2 is the Gaussian constant and i, j, k, the combination of the numbers 1, 2, 3, of the three masses. Furthermore, the equilibrium of the centrifugal forces at the barycenter 0 yields the three equations

$$m_i^2s_i^2\mu^4 = m_j^2s_j^2 + m_k^2s_k^2 + 2m_jm_ks_js_r\cos\varphi_i \tag{6-45}$$

φ_i can be eliminated by means of the geometrical relation

$$s_j^2 + s_k^2 - 2s_js_k\cos\varphi_i = r_{jk}^2$$

and Eq. (6-45) becomes, omitting μ^4,

$$s_j^2m_j^2 + s_k^2m_k^2 - s_i^2m_i^2 = (r_{jk}^2 - s_j^2 - s_k^2)m_jm_k \tag{6-46}$$

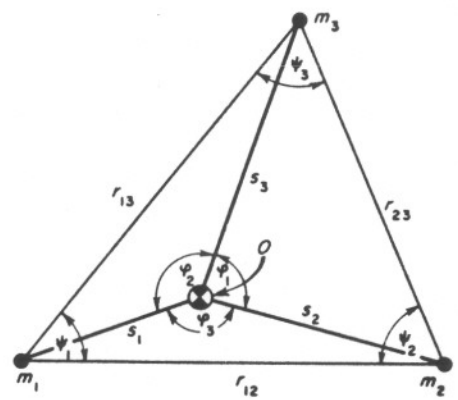

FIG. 6-2 Lagrange's Periodic Solution for Equidistant Particles.

Inserting, respectively, $i = 1, 2, 3$, $j = 2, 3, 1$, $k = 3, 1, 2$, one obtains three equations of the type (6-46). Adding these equations, one obtains

$$s_1^2m_1^2 + s_2^2m_2^2 + s_3^2m_3^2 = \frac{r_{23}^2m_2m_3 + r_{13}^2m_1m_3 + r_{23}^2m_2m_3}{\sum_1^3 m} \tag{6-47}$$

Writing Eq. (6-44) out for the three masses and substituting the resulting expressions for s_1, s_2, s_3, into Eq. (6-47) and rearranging yields the general equation

$$k^4m_1m_2m_3(r_{23}^{-4} + r_{12}^{-4} + r_{13}^{-4} + 2r_{12}^{-2}r_{13}^{-2}\cos\psi_1 + 2r_{13}^{-2}r_{23}^{-2}\cos\psi_2$$

$$+ 2r_{23}^{-2}r_{12}^{-2}\cos\psi_3) = \sum m_im_j\left(k^4\frac{\sum_1^3 m}{r_{ij}^4} - \frac{r_{ij}^2\mu^4}{\sum_1^3 m}\right) \tag{6-48}$$

If this equation is to hold true for arbitrary mass ratios $m_1:m_2:m_3$, then the expressions in the parentheses must become zero. In other words,

$$\frac{k^4}{\mu^4}\left(\sum_1^3 m\right)^2 = r_{ij}^6 \tag{6-49}$$

whence

$$r_{12} = r_{23} = r_{13} = r$$

and therefore in the other term in parentheses

$$\psi_1 = \psi_2 = \psi_3 = 60°$$

The mass triangle is therefore equilateral. The length of its side follows from

$$r = \left(\frac{k^2\sum_1^3 m}{\mu^2}\right)^{1/3} \tag{6-50}$$

Thus, by knowing the masses m_1, m_2, m_3 and specifying arbitrarily the angular velocity of rotation μ, the triangle connecting the equidistant masses is determined. In order to find the barycentric distance s_i of the mass m_i one writes Eq. (6-44) in its now simplified form, taking the square root and utilizing Eq. (6-50) to eliminate μ^2. This finally yields

$$s_i = r\frac{(m_j^2+m_k^2+m_jm_k)^{1/2}}{\sum_1^3 m} \tag{6-51}$$

The azimuthal acceleration of the mass m_i follows from the simplified Eq. (6-44) and (6-50) to be

$$s_i\mu^2 = \frac{k^2(m_j^2+m_k^2+m_jm_k)^{3/2}}{r^2\left(\sum_1^3 m\right)^2} \tag{6-52}$$

$$((k^2\mathrm{lb}^{6/2})/r^2\mathrm{lb}^2 = L^3T^{-2}\mathrm{lb}^{-1}\mathrm{lb}^3/L^2\mathrm{lb}^2 = L/T^2 = \text{acceleration})$$

The body moves as though it were attracted by the barycenter with a Newtonian force. Therefore the body describes a conic with the barycenter in one focus. Equation (6-51) apparently holds true for every body in the system.

In the case of the collinear particle solution, $\psi=0$, the values of r cannot be equal. For $\psi=0$ Eq. (6-44) becomes

$$s_i^2 = k^2\left(\frac{m_j}{r_{ij}^2}+\frac{m_k}{r_{ik}^2}\right) \tag{6-53}$$

or, if s again designates the respective distances from the barycenter,

$$s_i\mu^2 = k^2\left[\frac{m_j}{(s_j-s_i)^2}+\frac{m_k}{(s_k-s_i)^2}\right] \tag{6-54}$$

It is assumed that $s_1<s_2<s_3$, which means no general limitation. From these three equations one can develop a fifth degree equation in $\zeta=(s_3-s_2)/(s_2-s_1)$,

$$m_1\zeta^2[(1+\zeta^3)-1]+m_2(1+\zeta)^2(\zeta^3-1)+m_3[\zeta^3-(1-\zeta)^3] = 0 \tag{6-55}$$

This equation has one real and positive root. Of particular interest is the case where m_1 is much larger than m_2 and m_3, as in the case of the Sun and planets.

First assume that the outer mass is the large mass. Then the approximate value of the root is

$$\zeta \approx \left(\frac{m_2+m_3}{3m_1}\right)^{1/3} \tag{6-56}$$

This yields the following approximate barycentric distances for m_2 and m_3:

$$\frac{s_2}{s_1} \approx \frac{m_1}{m_2+m_3} \tag{6-57}$$

$$\frac{s_3}{s_1} \approx \frac{m_1}{m_2+m_3}(1+\zeta) \tag{6-58}$$

Since ζ is very small if $m_1 \gg m_2$ and $m_1 \gg m_3$, it follows $s_2 \approx s_3$, whence

$$\frac{s_3-s_2}{s_2} \approx \frac{s_3-s_2}{s_3} \approx \left(\frac{m_2+m_3}{3m_1}\right)^{1/3} \tag{6-59}$$

The libration points for m_2 and m_3 are close together, but their distance from m_1 is large as was to be expected (Fig. 6-3 (a)); r_1 follows from Eq. (6-54) after ω has been specified; $r_{12}=s_1+s_2$, $r_{13}=s_1+s_3$.

Now, assume that the large mass is m_2, i.e., lies between the smaller masses m_1 and m_3. If $m_1=m_3$, the distances s_1 and s_3 are equal and $s_2=0$. In this case (Fig. 6-3 (b)),

$$\frac{s_1}{s_2} = \frac{m_2}{m_1-m_3} \tag{6-60}$$

$$\frac{s_3}{s_2} = \frac{s_1}{s_2}\left(1+\frac{5}{12}\frac{m_1-m_3}{m_2}\right)-1 \tag{6-61}$$

$$\frac{r_{12}}{s_1} = 1+\frac{m_1-m_3}{m_2} \tag{6-62}$$

$$\frac{r_{23}}{s_1} = 1+\frac{5}{12}\frac{m_1-m_3}{m_2} \tag{6-63}$$

$$\frac{r_{23}}{r_{12}} = 1-\frac{7}{12}\frac{m_1-m_3}{m_2} \tag{6-64}$$

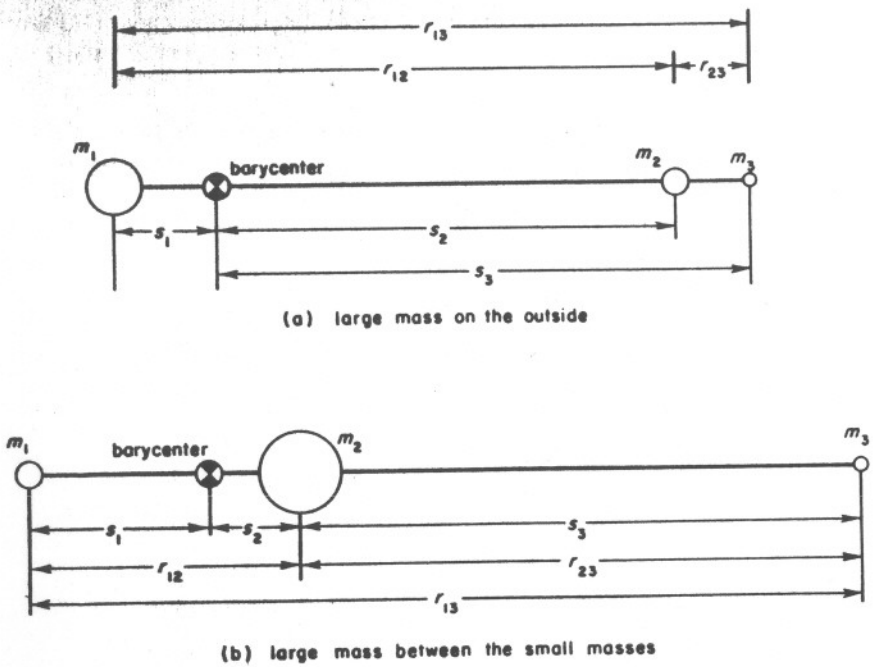

FIG. 6-3 Three-Body Solution for Collinear Particles.

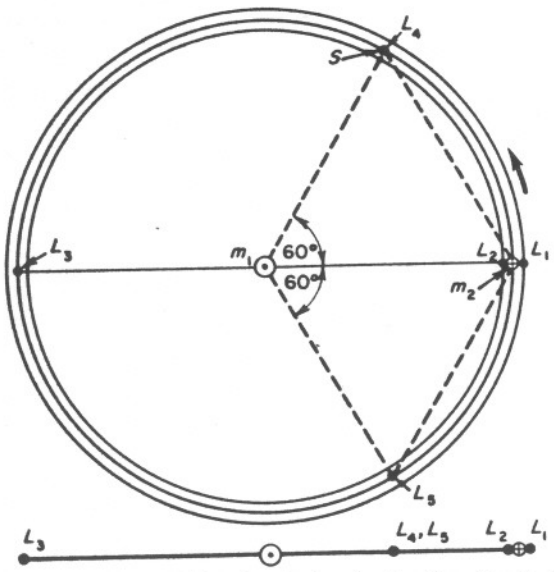

FIG. 6-4 Position of the Libration Points in the Sun-Earth System.

Assume, finally, that one mass is negligibly small, dm, and one mass, M, is much larger than the remaining finite mass. Take the distance between the two finite masses as D and the distance between M and dm as R. Then one has the following collinear solutions (Fig. 6-3):

(I) $m_2 = dm, \quad m_1 = M, \quad s_1 = 0, \quad s_2 = \rho_1, \quad s_3 = R$

From Eq. (6-59) it follows then (libration point No. 1),

$$\frac{\rho_1}{R} = \left[1 - \left(\frac{m_3}{3M}\right)^{1/3}\right] = L_1 \tag{6-65}$$

(II) $m_3 = dm, \quad m_1 = M, \quad s_1 \approx 0, \quad s_2 \approx R, s_3 \approx \rho_2$

Eq. (6-59) then yields the second libration point

$$\frac{\rho_2}{R} = \left[1 + \left(\frac{m_2}{M}\right)^{1/3}\right] = L_2 \tag{6-66}$$

(III) $m_2 = M, \quad m_3 = dm, \quad s_2 \approx 0, \quad r_{23} \approx \rho_3, \quad r_{12} \approx R.$

Eq. (6-64) then yields directly the third libration point

$$\frac{\rho_3}{R} = \left(1 - \frac{7}{12}\frac{m_1}{M}\right) = L_3 \tag{6-67}$$

These three libration points are shown for the Sun-Earth system in Fig. 6-4, together with the two equidistant libration points L_4 and L_5. Although L_1 and L_2 lie very close to the Earth by interplanetary standards, their distance from the Earth is nevertheless 813,300 n. mi. to either side; i.e., 4 times the distance to the Moon. L_3 does not lie exactly on the Earth's orbit (assumed to be circular) but slightly closer to the Sun. L_4 and L_5 are actually occupied, at least approximately, in the Sun-Jupiter system.[4] Two groups of at least 5 asteroids each, known collectively as the Trojan group are clustered around L_4 and L_5, respectively, moving 60° ahead and 60° behind Jupiter.

6-4 THE PLANAR RESTRICTED THREE-BODY PROBLEM

Because of the severe limitation in the analytical treatment of the three-body problem, with more or less arbitrary mass ratios, a special problem defined by Jacobi is of particular theoretical and practical importance. Jacobi assumed that one of the three masses is negligibly small and that one of the two remaining finite masses is much larger than the other (e.g., Earth and Moon). The problem is to investigate the motion of the body of infinitesimal mass in the gravitational field of the two finite masses. Let the dominant mass be m_1; the other finite mass be m_2. The particle

[4] Cf. Par. 3-7.

will be designated by S. Let s_1 and s_2 be the distances of the large mass and the small mass, respectively, from the barycenter. The distance between the two masses m_1 and m_2 is $D = s_1 - s_2$. Assume a fixed coordinate system x, y with the origin in the barycenter (Fig. 6-5). The

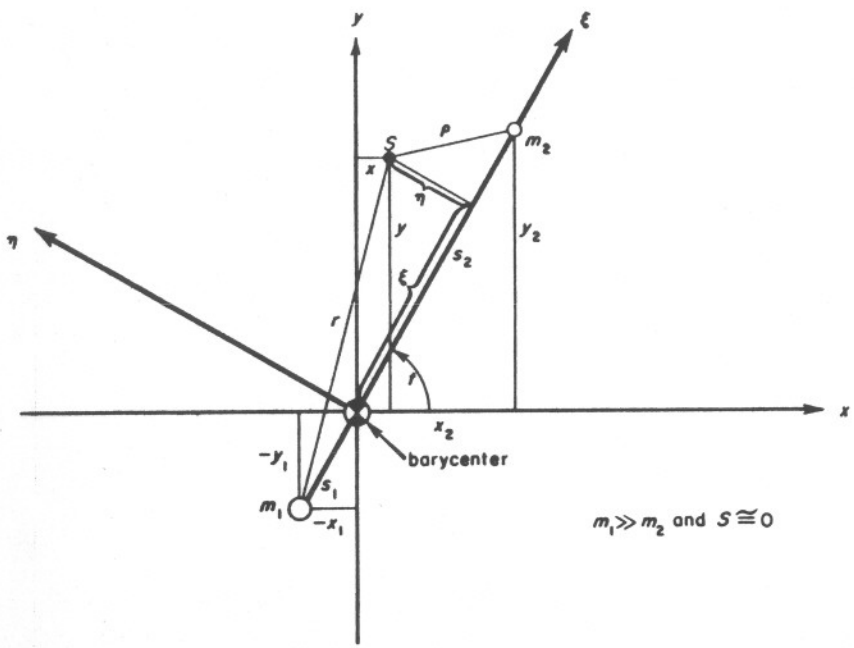

FIG. 6-5 Planar Restricted 3-Body Problem in Rectangular Fixed and Rectangular Rotating (Jacobian) Coordinates.

components of acceleration of the particle S in the x- and y-direction are then given by

$$\left.\begin{aligned} \ddot{x} &= -K_1\frac{x-x_1}{r^3} - K_2\frac{x-x_2}{\rho^3} \\ \ddot{y} &= -K_1\frac{y-y_1}{r^3} - K_2\frac{y-y_2}{\rho^3} \end{aligned}\right\} \tag{6-68}$$

where $K_1 = k^2m_1$, $K_2 = k^2m_2$, as before. The cartesian coordinates of m_1 and m_2 can be expressed in terms of a_1, a_2 and the angle μt between the connecting line m_1m_2 and the x-axis, where it is assumed that the whole system m_1m_2 revolves about the barycenter at constant angular velocity μ so that the instantaneous angle is μt.

Thus,

$$\left.\begin{aligned} x_1 &= s_1\cos\mu t, \quad y_1 = s_1\sin\mu t \\ x_2 &= s_2\cos\mu t, \quad y_2 = s_2\sin\mu t \end{aligned}\right\} \tag{6-69}$$

whence

$$\left.\begin{aligned} r^2 &= (x-s_1\cos\mu t)^2 + (y-s_1\sin\mu t)^2 \\ \rho^2 &= (x-s_2\cos\mu t)^2 + (y-s_2\sin\mu t)^2 \end{aligned}\right\} \tag{6-70}$$

From Eq. (4-260) and (4-262),

$$\mu = \sqrt{\frac{K_1+K_2}{(a_1+s_2)^3}} = \sqrt{\frac{K_1+K_2}{D^3}} \tag{6-71}$$

The equations of motion obtained in relation to a fixed coordinate system can be greatly simplified by introducing a rotating coordinate system ξ, η. This coordinate system is shown in Fig. 6-5 where the ξ-axis coincides with the radius vector m_1m_2 and the coordinate system therefore rotates with the angular velocity μ. The coordinates have been introduced by Jacobi and therefore are sometimes referred to as *Jacobian coordinates*.

In deriving the equations of motion of the restricted three-body problem in the rotating coordinate system, let it first be assumed that ξ does not necessarily coincide with the line m_1m_2 and that the angular velocity of the coordinate system is generally $\dot{\varphi}$, not necessarily constant. Then, applying Lagrange's equations (Par. 6-2 (a) and (b)) the two-dimensional equations of motion of the infinitesimal mass S are easily derived in their most general form.

It is

$$\begin{aligned} x &= \xi\cos\varphi - \eta\sin\varphi \\ y &= \xi\sin\varphi + \eta\cos\varphi \\ q_1 &= \xi, \quad q_2 = \eta, \quad \dot{q}_1 = \dot{\xi}, \quad \dot{q}_2 = \dot{\eta} \\ \dot{x} &= \dot{\xi}\cos\varphi - \xi\sin\varphi\dot{\varphi} - \dot{\eta}\sin\varphi - \eta\cos\varphi\dot{\varphi} \\ \dot{y} &= \dot{\xi}\sin\varphi + \xi\cos\varphi\dot{\varphi} + \dot{\eta}\cos\varphi - \eta\sin\varphi\dot{\varphi} \end{aligned}$$

The kinetic energy of the particle is $T = (m/2)v^2 = v^2/2$, per unit mass of the body S. Since in the planar case $v^2 = \dot{x}^2 + \dot{y}^2$, one obtains

$$T = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) = \tfrac{1}{2}[\dot{\xi}^2 + \dot{\eta}^2 + 2\dot{\varphi}(\xi\dot{\eta} - \dot{\xi}\eta) + \dot{\varphi}^2(\xi^2+\eta^2)] \tag{6-72}$$

$$U = -U(\xi, \eta)$$

First coordinate:

$$-\frac{\partial U}{\partial q_1} = -\frac{\partial U}{\partial \xi}$$

$$\frac{\partial T}{\partial \xi} = \dot{\varphi}\dot{\eta} + \dot{\varphi}^2\xi$$

$$\frac{\partial T}{\partial \dot{\xi}} = \dot{\xi} - \dot{\varphi}\eta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) = \ddot{\xi} - \ddot{\varphi}\eta - \dot{\varphi}\dot{\eta}$$

First equation of motion:

$$\ddot{\xi}-\ddot{\varphi}\eta-2\dot{\varphi}\dot{\eta}-\dot{\varphi}^2\xi-\frac{\partial U}{\partial \xi} = 0$$

Second coordinate:

$$-\frac{\partial U}{\partial q_2} = -\frac{\partial U}{\partial \eta}$$

$$\frac{\partial T}{\partial \eta} = -\dot{\varphi}\dot{\xi}+\dot{\varphi}^2\eta$$

$$\frac{\partial T}{\partial \dot{\eta}} = \dot{\eta}+\dot{\varphi}\xi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\eta}}\right) = \ddot{\eta}+\ddot{\varphi}\xi+\dot{\varphi}\dot{\xi}$$

Second equation of motion:

$$\ddot{\eta}+\ddot{\varphi}\xi+2\dot{\varphi}\dot{\xi}-\dot{\varphi}^2\eta-\frac{\partial U}{\partial \eta} = 0$$

The two general equations of motion are thus:

$$\left.\begin{aligned}\ddot{\xi}-\ddot{\varphi}\eta-2\dot{\varphi}\dot{\eta}-\dot{\varphi}^2\xi &= \frac{\partial U}{\partial \xi}\\ \ddot{\eta}+\ddot{\varphi}\xi+2\dot{\varphi}\dot{\xi}-\dot{\varphi}^2\eta &= \frac{\partial U}{\partial \eta}\end{aligned}\right\} \tag{6-73}$$

The first simplification is to assume $\dot{\varphi}=\mu=$ constant. Then Eq. (6-73) become

$$\left.\begin{aligned}\ddot{\xi}-2\mu\dot{\eta}-\mu^2\xi &= \frac{\partial U}{\partial \xi}\\ \ddot{\eta}+2\mu\dot{\xi}-\mu^2\eta &= \frac{\partial U}{\partial \eta}\end{aligned}\right\} \tag{6-74}$$

A second simplification is to put $\mu=1$. In this case, $2\pi/T=\mu=1$ or the unit of the period of revolution of T is 2π (seconds, hours, days or whichever unit is used to measure time); and the distance between m_1 and m_2 becomes $D=(K_1-K_2)^{1/3}$. Although these units are inconvenient to use in practical computations, they permit an especially simple form of the differential equations of motion. This form will be used to derive the Jacobian integral, the only known integral of these equations. Thus, for $\mu=1$ one can write

$$\left.\begin{aligned}\ddot{\xi}-2\dot{\eta} &= \frac{\partial U}{\partial \xi}+\xi = \frac{\partial \Phi}{\partial \xi}\\ \ddot{\eta}+2\dot{\xi} &= \frac{\partial U}{\partial \eta}+\eta = \frac{\partial \Phi}{\partial \eta}\end{aligned}\right\} \tag{6-75}$$

where

$$\Phi = \tfrac{1}{2}(\xi^2+\eta^2)+U \tag{6-76}$$

$$-U = \frac{K_1}{r}+\frac{K_2}{\rho} \tag{6-77}$$

Multiplying the first Eq. (6-75) by $2\dot{\xi}$, the second by $2\dot{\eta}$, and adding, one obtains

$$2(\ddot{\xi}\dot{\xi}+\ddot{\eta}\dot{\eta}) = 2\left(\frac{\partial \Phi}{\partial \xi}\frac{\partial \xi}{\partial t}+\frac{\partial \Phi}{\partial \eta}\frac{\partial \eta}{\partial t}\right) = 2\dot{\Phi}$$

This equation has been integrated by Jacobi to

$$\dot{\xi}^2+\dot{\eta}^2 = 2\Phi-C \tag{6-78}$$

The result is called the *Jacobian integral* and the constant C the *Jacobian constant*. The integral can be written in the form

$$\dot{\xi}^2+\dot{\eta}^2 = \xi^2+\eta^2+2U \tag{6-79}$$

In numerical computations Eq. (6-79) can serve as a convenient means for checking the accuracy of results. Eq. (6-72) can now be written in the form

$$T = \tfrac{1}{2}(\dot{\xi}^2+\dot{\eta}^2)+(\xi\dot{\eta}-\dot{\xi}\eta)+\tfrac{1}{2}(\xi^2+\eta^2)$$

and the Lagrangian

$$L = \tfrac{1}{2}(\dot{\xi}^2+\dot{\eta}^2)+(\xi\dot{\eta}-\dot{\xi}\eta)+\tfrac{1}{2}(\xi^2+\eta^2)+U$$

The integral of Jacobi

$$\tfrac{1}{2}(\dot{\xi}^2+\dot{\eta}^2)-[\tfrac{1}{2}(\xi^2+\eta^2)+U] = \text{constant}$$

is an expression of the conservation of relative energy with respect to the barycenter. The term $\frac{1}{2}(\xi^2+\eta^2)$ is the centrifugal force function due to the uniform ($\mu=$ const.) rotation of the system; and $(\xi\dot{\eta}-\dot{\xi}\eta)$ represents the coriolis forces. Since they are not an energy term they do not occur in the Jacobian integral.

Equations (6-74) with the simplification $\dot{\varphi}=\mu$ are accurate enough for many restricted body computations; however, it remains to establish the solutions for the instantaneous gravitational acceleration $\partial U/\partial \xi$ and $\partial U/\partial \eta$. As to the first

$$U = \frac{K_1}{r}+\frac{K_2}{\rho}$$

Then

$$\frac{\partial U}{\partial \xi} = -\frac{K_1}{r^2}\frac{\partial r}{\partial \xi}-\frac{K_2}{\rho^2}\frac{\partial \rho}{\partial \xi}$$

Let ξ, η be the coordinates of the particle S; ξ_1, η_1 those of m_1; and ξ_2, η_2 those of m_2. Then

$$r = [(\xi-\xi_1)^2+(\eta-\eta_1)^2]^{1/2}$$

$$\frac{\partial r}{\partial \xi} = [(\xi-\xi_1)^2+(\eta-\eta_1)^2]^{-1/2}(\xi-\xi_1) = \frac{\xi-\xi_1}{r}$$

$$\rho = [(\xi-\xi_2)^2+(\eta-\eta_2)^2]^{1/2}$$

$$\frac{\partial \rho}{\partial \xi} = \frac{\xi-\xi_2}{\rho}$$

whence

$$\frac{\partial U}{\partial \xi} = -\frac{K_1}{r^3}(\xi-\xi_1)-\frac{K_2}{\rho^3}(\xi-\xi_2) \tag{6-80}$$

Analogously one finds

$$\frac{\partial U}{\partial \eta} = -\frac{K_1}{r^3}(\eta-\eta_1)-\frac{K_2}{\rho^3}(\eta-\eta_2) \tag{6-81}$$

If the ξ-axis coincides and rotates with the line m_1m_2, then $\eta_1=\eta_2=0$ and $\xi_1=-s_1$, $\xi_2=s_2$.

The final equations of motion of the planar restricted three-body problem in Jacobian coordinates are therefore (combining Eq. (6-74), (6-80), (6-81)):

$$\left.\begin{aligned}
\ddot{\xi} &= 2\mu\dot{\eta}+\mu^2\xi-\frac{K_1}{r^3}(\xi+s_1)+\frac{K_2}{\rho^3}(\xi-s_2)\\
\ddot{\eta} &= -2\mu\dot{\xi}-\eta\left[\frac{K_1}{r^3}+\frac{K_2}{\rho^3}+\mu^2\right]\\
r^2 &= (\xi+s_1)^2+\eta^2\\
\rho^2 &= (\xi-s_2)^2+\eta^2\\
\mu^2 &= \frac{K_1+K_2}{(s_1+s_2)^3} = \frac{K_1+K_2}{D^3}
\end{aligned}\right\} \tag{6-82}$$

In the two equations of motion the individual terms have the following significance:

$2\mu\dot{\eta}$, $2\mu\dot{\xi}$: Coriolis acceleration due to radial motion in a rotating system;
$\mu^2\xi$, $\mu^2\eta$: Centrifugal acceleration due to angular motion about the barycenter.
The terms containing K_1 and K_2: The components of gravitational acceleration in ξ and η of the two finite masses m_1 and m_2.

With the above equations of motion the flight path of a particle S between two masses $m_1>m_2$ can be computed by means of numerical integration. The computation is laborious and time-consuming, unless done very crudely. However, in modern digital computers, means have become available to reduce the accurate numerical integration of such a path from several months to a few hours, *after* the problem has been programmed into the computer. Equations (6-82) are in a suitable form

for computing machines and have been used for computing the path of space vehicles between Earth and Moon.[5]

It should finally be mentioned that in those cases where the particle comes extremely close to one of the finite masses or even collides with it, a transformation of coordinates and time may be found useful, a methodology first proposed by Thiele at the end of the last century. Instead of ξ and η the coordinates E and F are introduced and the time t is replaced by a parameter ψ and becomes a function of the distances r and

$$\left.\begin{aligned}
\xi &= \sinh F\cos E\\
\eta &= -\sinh F\sin E\\
dt &= r\rho d\psi
\end{aligned}\right\} \tag{6-83}$$

Thus

$$\left.\begin{aligned}
\tan E &= -\eta/\xi\\
\sinh F &= \xi\sec E
\end{aligned}\right\} \tag{6-84}$$

Equations (6-83) can be inserted in (6-82). It is seen that if either r or ρ becomes very small, the introduction of the parameter ψ has the effect of decreasing the unit of time, as if the motion of the particle were measured not in ft/sec, but in ft per 10^{-2} sec or 10^{-3}, etc., the negative exponent increasing as the particle comes closer to the respective finite mass. For electrical computer calculations, however, it is not necessary to carry out the Thiele transformation.

6-5 PERIODIC SOLUTIONS NEAR THE LIBRATION POINTS

By developing each of the functions $\partial\Phi/\partial\xi$ and $\partial\Phi/\partial\eta$ in the derivation of the Jacobi integral into a series and restricting the analysis to periodic orbits which lie in the immediate vicinity of a given point (ξ, η), where "immediate vicinity" means $\partial\Phi/\partial\xi=\partial\Phi/\partial\eta=0$, one can show that the points (ξ, η) in whose vicinity periodic orbits become possible (libration points) have the following coordinates:

$$\left.\begin{aligned}
L_1:&\quad \frac{\xi_1}{s_1+s_2} = -\frac{1}{1+m}+\left(\frac{m}{3}\right)^{1/3}-\tfrac{1}{3}\left(\frac{m}{3}\right)^{2/3};\quad \eta_1 = 0\\
L_2:&\quad \frac{\xi_1}{s_1+s_2} = -\frac{1}{1+m}-\left(\frac{m}{3}\right)^{1/3}-\tfrac{1}{3}\left(\frac{m}{3}\right)^{2/3};\quad \eta_2 = 0\\
L_3:&\quad \frac{\xi_3}{s_1+s_2} = \frac{m}{1+m}+1-\tfrac{7}{12}m+\tfrac{7}{12}m^2;\quad \eta_3 = 0\\
L_4:&\quad \frac{\xi_4}{s_1-s_2} = -\tfrac{1}{2}\frac{1-m}{1+m};\quad \frac{\eta_4}{s_1+s_2} = \frac{\sqrt{3}}{2}\\
L_5:&\quad \frac{\xi_5}{s_1+s_2} = -\tfrac{1}{2}\frac{1-m}{1+m};\quad \frac{\eta_4}{s_1+s_2} = -\frac{\sqrt{3}}{2}
\end{aligned}\right\} \tag{6-85}$$

[5] Cf. *Space Flight*, Vol. 2, *Dynamics*.

where m is the mass ratio m_2/m_1, m_1 being the dominant mass (Fig. 6-4). The term $\sqrt{3}/2$ for η_4 and η_5 results from the fact that the triangle m_1m_2S is equilateral; hence the height of point s "above" the base line m_1m_2 is equal to $\sqrt{(m_1S)^2-(m_2S)^2/4}$ and, since both lines are equal, $(mS/2)\sqrt{3}$; since the triangle is equilateral $mS=s_1+s_2$, the sum of the barycentric distances of m_1 and m_2. In other words, these libration points coincide with those points in which S would have to be in the case of an exact Lagrange solution of the general three-body problem (Par. 6-3). The libration points L_4 and L_5 form an equilateral triangle with the line m_1m_2 (Fig. 6-4). The libration points L_1 through L_3 lie on the line m_1m_2. The distance $\rho=m_2S$ of L_1 and L_2 which lie closest to m_2 are:

$$L_1: \quad \frac{\rho_1}{s_1+s_2} = \left(\frac{m}{3}\right)^{1/3} + \frac{1}{3}\left(\frac{m}{3}\right)^{2/3} \tag{6-86}$$

$$L_2: \quad \frac{\rho_2}{s_1+s_2} = \left(\frac{m}{3}\right)^{1/3} - \frac{1}{3}\left(\frac{m}{3}\right)^{2/3} \tag{6-87}$$

The distance of L_3 from m_1 is given by

$$L_3: \quad \frac{r_3}{s_1+s_2} = 1-\tfrac{7}{12}m+\tfrac{7}{12}m^2 \tag{6-88}$$

The motion of the particle S is called *periodic* if its position coordinates q_k and its velocity coordinates $\dot{q}_k$ relative to the uniformly rotating system m_1m_2 are equiperiodic functions of time, i.e., are repeated at constant intervals. Each of these periodic orbits is a closed curve within the rotating system. The libration points are loci about which S can oscillate in periodic orbits or which represent asymptotic points or boundary points of groups of periodic orbits.

In order to understand the importance of the periodic solutions, one must remember that the only available mathematical method for the three-body problem or restricted three-body problem is that of the numerical integration. Although this method permits of attaining practically any desired degree of accuracy, it cannot be used to predict the motion of the particle for any time, either before or after the period computed, i.e., unless the motion can be shown to be of periodic character. It is, therefore, important to find out whether a perturbed motion is of periodic character. For this reason, as many as possible, if not all, periodic solutions should be known. Since space flight will introduce a great number of new celestial mechanical "situations" it is probable that the theory of the periodic solutions will be an important field in analytical astronautics. Although it is impossible here to treat this subject in detail, an attempt is made to survey some of the fundamental aspects.

The investigation of periodic solutions was started during the final

decade of the last century by Darwin, Thiele, and Burran.[6] They made attempts to compute numerically a number of cases of the restricted problem so that a survey over several classes of periodic solutions would be obtained. Darwin chose $m_1/m_2=10$ and 11, Thiele $m_1/m_2=1$. The most thorough and systematic investigations, however, were conducted at the Copenhagen Observatory under the leadership of E. Strömgren and his son, B. Strömgren. As a result of these efforts during the first three decades of this century, the variety of possible periodic solutions is nearly completely known, at least in principle.[7] Subsequently, the main classes of periodic solutions are surveyed, based on E. Strömgren.[7] The mass ratio in all cases is $m_1/m_2=1$. The particle S has an initial position on the line m_1m_2. For each such initial position, the velocity normal to the line m_1m_2 must be found which makes the resulting orbit a closed curve. If the distance m_1m_2 is given, the constant angular velocity of the connecting line is specified, and the mass ratio m_1/m_2 is known, then one finds for each initial position one or more velocities leading to periodic orbits. All velocities associated with a given initial position are summarized in a given class. There exist at least 15 such classes which can be grouped as follows:

A. *Periodic orbits about both masses* m_1 and m_2: Periodic orbits which are retrograde in the rotating as well as absolute coordinate system (retrograde with respect to the rotation of the m_1m_2 axis, assumed to be counterclockwise). Orbits are shown in Fig. 6-6. At great distances from m_1 and m_2 these orbits are circular. As their distance decreases they become increasingly flat. Finally, orbit 4 moves at infinite velocity on a straight line between m_1 and m_2.

Another class contains orbits which are retrograde in the rotating coordinate system only (Fig. 6-7). At great distances the orbits approach circularity about the barycenter of m_1 and m_2. Orbits which are closer form loops in the vicinity of L_4 and L_5. Gradually these loops contract about L_4 and L_5 (2) and finally end asymptotically in these two libration points (3).

In both classes the orbital velocity is very small at great distances and increases as the orbits lie closer to m_1 and m_2.

B. *Periodic orbits about one of the two masses.* Since in the present survey $m_1=m_2$, the orbits are valid for either mass.

One class shown in Figure 6-8 contains retrograde orbits. First there exist circular orbits of great velocity in the vicinity of the mass m_2. At greater distances the orbits become oval (2) and kidney-shaped (3). The extreme case in this direction is an *ejection orbit* (4) where the particle is

[6] *Enzyklopaedie der Mathematischen Wissenschaften*, Vol. VI, 2-, Artikel 19, p. 967.

[7] E. Strömgren in *Ergebnisse der Exakten Naturwissenschaften*, Vol. IV, p. 233, Berlin, Springer, edit. (1925).

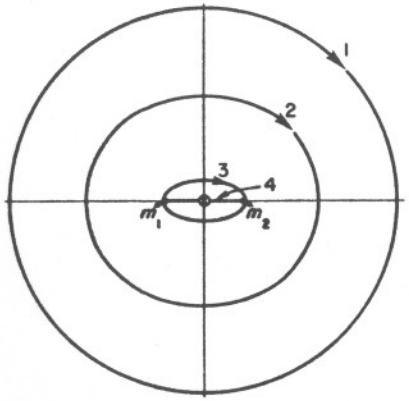

FIG. 6-6 Retrograde Periodic Orbits about Masses m_1 and m_2 in the Restricted 3-Body Problem.

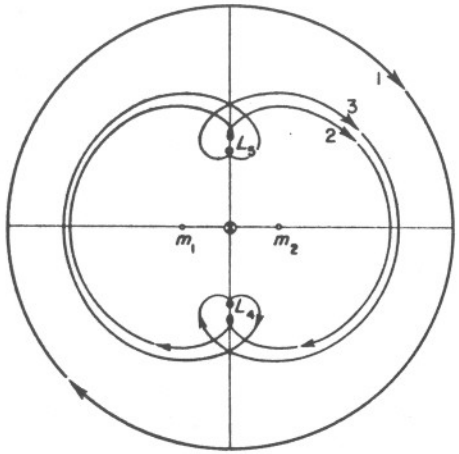

FIG. 6-7 Periodic Orbits about Masses m_1 and m_2 in the Restricted 3-Body Problem.

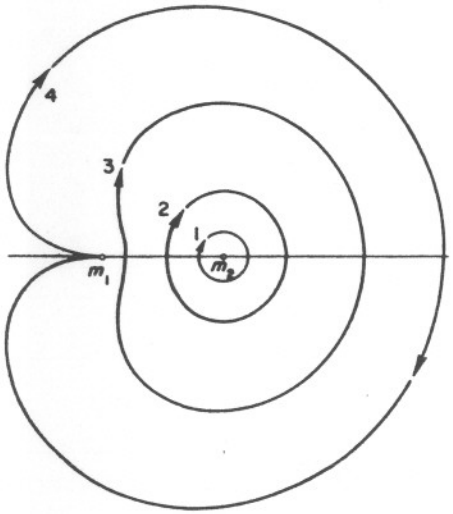

FIG. 6-8 Retrograde Periodic Orbits about Mass m_2 in the Restricted 3-Body Problem.

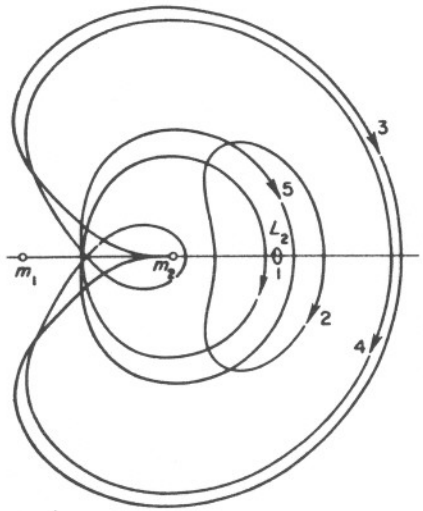

FIG. 6-9 Retrograde Periodic Orbits about Libration Point L_2 in the Restricted 3-Body Problem.

ejected from one mass at infinite velocity and returns to its origin after one revolution.

C. *Periodic orbits about a libration point.* Two classes of orbits involve a retrograde periodic motion about L_2 (or L_3), shown in Fig. 6-9. First

the particle can be assumed at rest in L_2; then it can move in elliptic orbits about L_2 (1). At greater distances from L_2 there follow kidney-shaped orbits (2) and finally ejection orbits (3) which begin and terminate in m_2 (or m_1) at infinite velocity. Then follow orbits with a loop around m_2 (or m_1), such as (4). Gradually the loop becomes larger and the orbit smaller (5) until the diameter of the loop equals that of the orbit. Thereafter the development is reversed, the orbit becomes the loop and vice versa and conditions finally lead to a particle at rest in L_2 (initial condition).

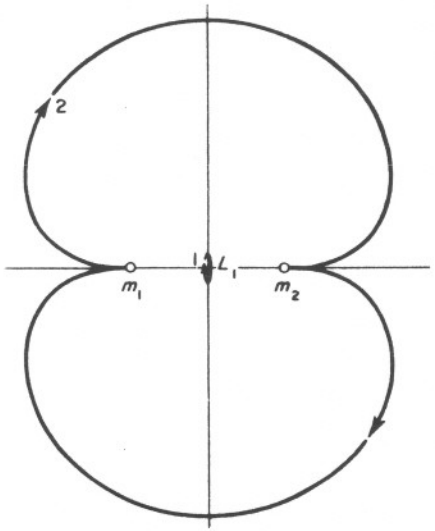

FIG. 6-10 Retrograde Periodic Orbits about Libration Point L_1 in the Restricted 3-Body Problem.

A third class begins with elliptic orbits about L_1 (Fig. 6-10) followed by "Mae West"-shaped orbits which finally degenerate into an ejection orbit (2). These are probably followed by orbits which describe loops about m_1 and m_2.

6-6 THE NON-PLANAR RESTRICTED THREE-BODY PROBLEM

In the case of an inclination of the satellite's orbital plane with respect to the plane of the lunar orbit, one has

$$x = \xi \cos \varphi - \eta \sin \varphi$$

$$y = \xi \sin \varphi + \eta \cos \varphi$$

$$z = \zeta$$

The time differentials x and y are given in Par. 6-3, and $\dot{z}=\dot{\zeta}$. Given $v^2=\dot{x}^2+\dot{y}^2+\dot{z}^2$, the kinetic energy

$$\begin{aligned} T &= \tfrac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) \\ &= \tfrac{1}{2}[\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2+2\dot{\varphi}(\xi\dot{\eta}-\dot{\xi}\eta)+\dot{\varphi}^2(\xi^2+\eta^2)] \end{aligned} \tag{6-89}$$

The potential energy is $U=-U(\xi,\eta,\zeta)$. The Lagrangian is

$$L = T+U(\xi,\eta,\zeta)$$

In analogy to Eq. (6-73) one arrives at the general equations of motion

$$\left.\begin{aligned} \ddot{\xi}-\ddot{\varphi}\eta-2\dot{\varphi}\dot{\eta}-\dot{\varphi}^2\xi &= \frac{\partial U}{\partial \xi} \\ \ddot{\eta}+\ddot{\varphi}\xi+2\dot{\varphi}\dot{\xi}-\dot{\varphi}^2\eta &= \frac{\partial U}{\partial \eta} \\ \ddot{\zeta} &= \frac{\partial U}{\partial \zeta} \end{aligned}\right\} \tag{6-90}$$

Replacing $\dot{\varphi}=\mu=$ constant, it follows

$$\left.\begin{aligned} \ddot{\xi}-2\mu\dot{\eta}-\mu^2\xi &= \frac{\partial U}{\partial \xi} \\ \ddot{\eta}+2\mu\dot{\xi}-\mu\eta^2 &= \frac{\partial U}{\partial \eta} \\ \ddot{\zeta} &= \frac{\partial U}{\partial \zeta} \end{aligned}\right\} \tag{6-91}$$

Again it is

$$U = \frac{K_1}{r}+\frac{K_2}{\rho}$$

$$\frac{\partial U}{\partial \xi} = -\frac{K_1}{r^2}\frac{\partial r}{\partial \xi}-\frac{K_2}{\rho^2}\frac{\partial \rho}{\partial \xi}$$

where now

$$\left.\begin{aligned} r &= [(\xi-\xi_1)^2+(\eta-\eta_1)^2+\zeta^2]^{1/2} \\ \rho &= [(\xi-\xi_2)^2+(\eta-\eta_2)^2+\zeta^2]^{1/2} \end{aligned}\right\} \tag{6-92}$$

whence

$$\left.\begin{aligned} \frac{\partial U}{\partial \xi} &= -\frac{K_1}{r^3}(\xi-\xi_1)-\frac{K_2}{\rho^3}(\xi-\xi_2) \\ \frac{\partial U}{\partial \eta} &= -\frac{K_1}{r^3}(\eta-\eta_1)-\frac{K_2}{\rho^3}(\eta-\eta_2) \\ \frac{\partial U}{\partial \zeta} &= -\zeta\left(\frac{K_1}{r^3}+\frac{K_2}{\rho^3}\right) \end{aligned}\right\} \tag{6-93}$$

If the ξ-axis coincides and rotates with the line m_1m_2, one has $\eta_1=\eta_2=0$ and $\xi_1=s_1$, $\xi_2=s_2$, so that the final equations of motion for the non-planar case become

$$\left.\begin{aligned} \ddot{\xi} &= 2\mu\dot{\eta}-\mu^2\xi-\frac{K_1}{r^3}(\xi+s_1)+\frac{K_2}{\rho^3}(\xi-s_2) \\ \ddot{\eta} &= -2\mu\xi-\eta\left[\frac{K_1}{r^3}+\frac{K_2}{\rho^3}+\mu^2\right] \\ \ddot{\zeta} &= -\zeta\left[\frac{K_1}{r^3}+\frac{K_2}{\rho^2}\right] \\ r^2 &= (\xi+s_1)^2+\eta^2+\xi^2 \\ \rho^2 &= (\xi-s_2)^2+\eta^2+\zeta^2 \\ \mu^2 &= \frac{K_1+K_2}{(S_1+S_2)^3} = \frac{K_1+K_2}{D^3} \end{aligned}\right\} \tag{6-94}$$

6-7 SPECIAL PERTURBATIONS

The practical significance of the special perturbations as compared to the general perturbations has been discussed in the introductory paragraph of this chapter. The special perturbation method requires a continuous determination of the perturbed orbit by means of stepwise integration of the acceleration of the body under the influence of all attracting forces. In earlier times the great advantage of practically unlimited accuracy in orbit computation, offered by the method of special perturbations, could frequently not be utilized because of the large amount of computations required, especially if longer periods of time were considered. This hurdle was removed with the advent of the electronic computer.

Three principal methods for computing special perturbations have been worked out in the 18th and 19th centuries. The first, and in astronomical celestial mechanics the most important, is that of the *variation of elements*. It was developed by Lagrange[8] who applied the mathematical method known as "variation of the constants" to the computation of special perturbations. The second and third methods are summarily known as the method of the *variation of coordinates*. They are distinguished by using the polar coordinates in one case and cartesian coordinates in the other. Their development was stimulated in the course of the 19th century when an increasing number of small planets (asteroids) was discovered, establishing a need for methods which would require less computation than the variation of elements. That the use of coordinates offers this advantage

[8] *Sur les perturbations, que les comètes peuvent éprouver par l'action des planètes*, Œuvres, Vol. VI, 1785. Later reproduced by Laplace, *Mécanique céleste*, Vol. IV, Livres IX: *Sur les comètes*.

becomes immediately apparent if one remembers that an orbit (usually elliptic) is determined by six elements, while only three coordinates have to be calculated. It was also found that coordinate perturbations usually are smaller than the perturbations of the elements. Therefore, perturbations of higher order[9] can be neglected with less penalty in terms of accuracy. However, it is a basic disadvantage of the variation of coordinates that a computation of the perturbed elements from the perturbed coordinates is laborious. Therefore, if the elements must be known during the period of calculation, for instance, in order to compute the ephemeris and for purposes of orbit improvement, then the *variation of elements* is the preferred method. The variation of coordinates is used:

(a) in the absence of observations;
(b) when the orbit of the perturbed body approaches a parabola, but when it is not permissible or desirable to substitute a parabola for the actual orbit and to neglect perturbations of higher order (if these two steps are permissible, one can still, without too much computational effort, apply the variation of elements). Such cases occur in astronomy where long-periodic comets are concerned. During the short periods of visibility, their orbit is near-parabolic.

The use of cartesian coordinates is preferable in cases of a single passage through a perturbing force field. If the body returns periodically into the range of observation and was exposed to perturbations between the period of observability, then these periods can more easily be connected with each other if polar coordinates are used. Again, the concern in regard to identifying various periods of visibility as representing the same body is chiefly of astronomical interest, namely, in the case of long-periodic comets whose perihelion passages may be many decades or centuries apart. Such cases are of no significance in astronautics. However, knowledge of the polar coordinates is important for navigational purposes. It would not be possible for the crew of a space ship to determine its position in cartesian coordinates by means of stellar navigation.

6-8 VARIATION OF ELEMENTS

The principle of this method is to approximate sections of the actual (perturbed orbit) by a conic, i.e., an unperturbed orbit. Such an unperturbed orbit is designated as an *osculating conic* or *osculating orbit*. An osculating conic is therefore defined as a conic which coincides with a differential portion of the actual orbit.[10] Its elements are called *osculating elements*. The moment at which the body covers this portion of its orbit is called the *osculation epoch*. The statement that the osculating conic describes a portion of the actual orbit means that during the particular osculation epoch the corresponding osculating conic can be used to determine the position and velocity of the body with sufficient accuracy. Obviously, if during a given osculation epoch the perturbing force should disappear, then the respective conic would, from then on, represent the correct orbit of the body. By describing the actual orbit as a sequence of osculating conics, the perturbed orbit itself can be regarded as a conic, characterized by variable elements.

[9] *Perturbations of higher order* are defined as those of the second, third, etc., order. They are caused by the fact that the perturbation of the first order changes the original conic path of the perturbed body. Due to the change in orbit, the perturbation changes also from what it would have been if the body were not deviated previously. In many cases the orbital deviation, although significant, changes the acting perturbation force (i.e., the first order perturbation) so little that its adjustment (by means of the higher order perturbations) can be neglected.

6-9 LAGRANGE'S SPECIAL PERTURBATION EQUATIONS ACCORDING TO THE VARIATIONS OF ELEMENTS IN THE ELLIPTIC CASE

The simple central force field equations (no perturbation) are

$$\ddot{s}+\frac{K}{r^3}s_0 = 0, \quad (s = x, y, z) \tag{6-95}$$

where

$$K = k^2(M+m) \sim k^2M \tag{6-96}$$

M being the central mass, m the (negligible) mass of the body B, r_0 the radial distance from M to B, and x, y, z the coordinates of B in a cartesian system with the origin in M. The above three second-order differential equations, which require six integration constants, are satisfied by three equations defining x_0, y_0, z_0. For an elliptic orbit, let

$$\left.\begin{aligned} \mu &= \sqrt{K/a^3} \\ p &= a(1-e^2) \\ E-e\sin E &= \mu t_0+\sigma \\ \sigma &= -\mu T_P \quad \text{(cf. below)} \\ \tan\tfrac{1}{2}\eta &= \tan\tfrac{1}{2}E\sqrt{\frac{1+e}{1-e}} \\ r &= \frac{p}{1+e\cos\eta} \end{aligned}\right\} \tag{6-97}$$

[10] By definition, the orbit is not a conic but a more complicated, three-dimensional path which is not closed and which cannot be determined by closed integration of the equations of motion.

(*32*) A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, Princeton, 1947. (Mathematically very advanced.)

(*33*) P. Herget, *The Computation of Orbits*. Published by the author, Director of the Cincinnati Observatory, Cincinnati, Ohio, 1948.
(A "must" for everybody interested in actual computations in special and general perturbations.)

(*34*) W. M. Smart, *Celestial Mechanics*, Longmans, Green & Co., 1953.
(As a modern introduction to celestial mechanics for the graduate student, the book complements Moulton's work.)

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PRINCIPLES OF
GUIDED MISSILE DESIGN

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SPACE FLIGHT

I. ENVIRONMENT AND CELESTIAL MECHANICS

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