INTERPLANETARY TRAJECTORIES*

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The aim of this work is to solve the two-point "limit problem" within the gravitational field of the solar system. To make possible a concise mathematical formulation of the problem, a method is presented for the definition of a physical model "ad hoc" to whatever interplanetary trajectories must be studied. Picard's iterative method for constructing solutions, with an essential modification found empirically, is given and numerical results systematically obtained are presented.

INTRODUCTION

In recent years, considerable work has been done on the problem of determining interplanetary trajectories, both in the United States and abroad. At home, various n-body programs, which are a mixture of astronomical and numerical integration techniques, were developed. These programs can be used directly for the solution of the initial boundary value problem: "Given the position and velocity at a given instant, determine the corresponding trajectory." By application of a trial and error procedure these programs have also been used to solve numerically the boundary value problem we call the "limit problem": "Determine the trajectory that goes through two given points at two given instants." The mathematical foundations that justify this type of approach were given at

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the turn of the century by Paul Painleve. He showed that the singularities in the equations of celestial mechanics are always removable. From these thoughts sprang epoch-making papers by Levi-Civita and Sundman. Recently, Benedikt[1] has determined the technical value and limitations of Levi-Civita's work. My teacher, the late Professor G. D. Birkhoff,[2] elucidated Sundman's paper and made it accessible to a large public. However, from the point of view of scientific engineering, one is not concerned with "collision trajectories" in the sense of modern dynamics. Instead, it is the "limit problem" that has primary importance in technical applications.

The present work is divided into three parts. In Part 1 the physical model is defined. Although we have only dealt with trajectories from near the earth to the moon, the method outlined for the selection of the relevant planets, as well as the form of the corresponding differential equations, is completely general. This is, in spite of its simplicity, an essential novelty of this work.

Part 2 is divided into sections that, logically, are quite apart. Using ideas E. Picard[3] first published in 1893, the two-point limit problem is rigorously formulated in an integral form. Trying to use Picard's method of successive approximations for the actual construction of solutions, a cyclic process was devised - section 2.2 - by means of which solutions were obtained even for cases when the original iterative process is divergent. This scheme, as presented here, has no scientific value. However, if numerical results are viewed as phenomena, this cyclic scheme becomes a very interesting experimental tool, by use of which solutions can be obtained and the nature of the specific problem at hand can be analyzed. This work is not concerned with the determination of the general conditions of applicability of this scheme; but, it is hoped, this and related questions will be studied in a forthcoming paper.

In Part 3, numerical results of high accuracy are presented. The numerical analysis that should precede, or accompany, the construction of solutions is outlined in section 3.3.

1. PHYSICAL MODEL

1.1 GENERAL CONCEPTS

For any problem involving the calculation of interplanetary trajectories, and independent of what is the particular problem at hand, the first questions to resolve are what physical model and what coordinate system of reference should be used. In this work, a system of Cartesian coordinates at absolute rest - a Galilean system* - is assumed with center at the Sun, with respect to which another Cartesian system rigidly attached to it - and, naturally, Galilean - is defined and used as a system of reference. All the planets and the Sun are considered particles - points with finite mass - and our differential equations define the motion of a negligible mass in the time dependent gravitational field created by certain planets, selected as bearing mensurable influence upon the motion in question, in their known motions within the solar system. Since we take the results of astronomical work as physical data, our equations are very much simpler than those usually used in the so-called n-body programs; on the other hand, a selection of the planets that determine the motion must be made "a priori."

Let these assumptions be examined more closely. The equations of free dynamical systems are the analytic expressions for the symbolic relation,

$$\text{Inertial Terms} = \text{Acting Forces} \qquad (1)$$

Astronomers provide us with the positions of planets in the equatorial system - shown in Fig. 1 - versus time, with a given reference date. Since only gravitational forces are considered, and these are defined

*By definition, Galilean systems include also those moving without acceleration.

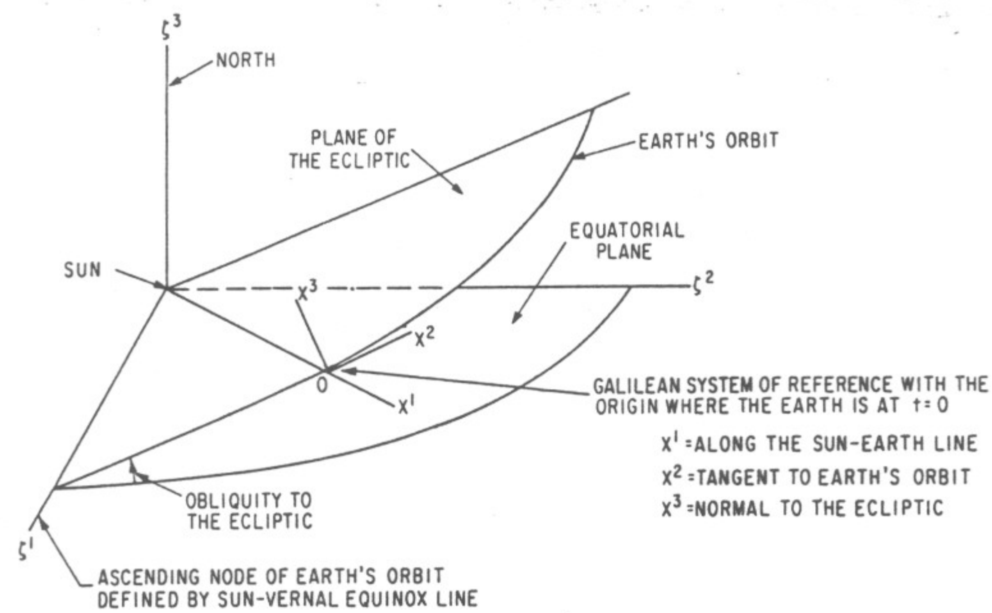

Fig.1 Definition of the Galilean System of Reference

completely by the relative position of the point masses, the right hand side of Eq. (1) will be written for our physical model with no "error", if the astronomical data is supposed to be "exact."

In Newtonian mechanics the existence of a system of coordinates is always assumed at absolute rest. Therefore, the error committed when writing the acceleration components on our Galilean system can only be examined with reference to motions known to exist and can only be determined with the accuracy with which those motions are quantitatively known. Let $\vec{\omega}_G$ and $\vec{\omega}_S$ be the angular velocity vectors of the motions of the center of mass, c. m., of the solar system about our galactic axis - the line at absolute rest - and of the Sun about the c. m. of the solar system, respectively, both $\vec{\omega}_G$ and $\vec{\omega}_S$ being obtained with the Sun as the reference point for the decomposition of the motions. If these motions are taken into account, the left-hand side of Eq. (1); i. e., the acceleration per unit mass, will be given by the vector.

$$\dot{\vec{V}} + (\vec{\omega}_G + \vec{\omega}_S) \times \vec{V}$$

where the local time-derivative, indicated by the "dot", is taken with respect to a system rigidly attached to the Sun.* Thence, the error committed on the left hand side of Eq. (1) due to the assumption that our system of reference is at absolute rest is given by

$$(\vec{\omega}_G + \vec{\omega}_S) \times \vec{V}$$

It is known that the solar system moves about our galactic axis with a periodic motion; let the period by $\overline{T}$. Neglecting $\vec{\omega}_S$, and assuming this motion rigid, for a vehicle moving at, say, 30,000 ft/sec and for an optimum relative position of its velocity vector, $\vec{V}$, and $\vec{\omega}_G$, the error in position will be less than 4,500 ft per year of flight time, for $\overline{T}$ equals 100 million years.

Let it be remarked that, although Einstein's gravitational tensor is independent of any specific system of coordinates, in order to write the differential equations of motion for the known integrable cases (Schwarzchild, Painlevé), a system of reference at absolute rest must also be explicitly assumed within the framework of the General Theory of Relativity.

1.2 PHYSICAL MODEL FOR EARTH-MOON TRAJECTORIES

The motion of a space vehicle - as a point mass - will be studied in the gravitational field created by the Sun, the Earth, and the Moon. The qualitative reasoning for this selection runs as follows: Although, for relatively short intervals of total flight time, the influence of the Sun may be negligible, there will always be a portion of the flight under the almost exclusive influence of the Sun. Moreover, to account for the presence of the Sun does not bring any extraneous complexity in the computations, for the distances from the vehicle to the Earth and to the Moon must be calculated using their coordinates in the equatorial system.

An example of the quantitative analysis that should precede the selection of planets will show how insufficient a reasoning such as that in the

*In Newtonian mechanics, "all the clocks are synchronized"; thus, only the relative motions need be considered.

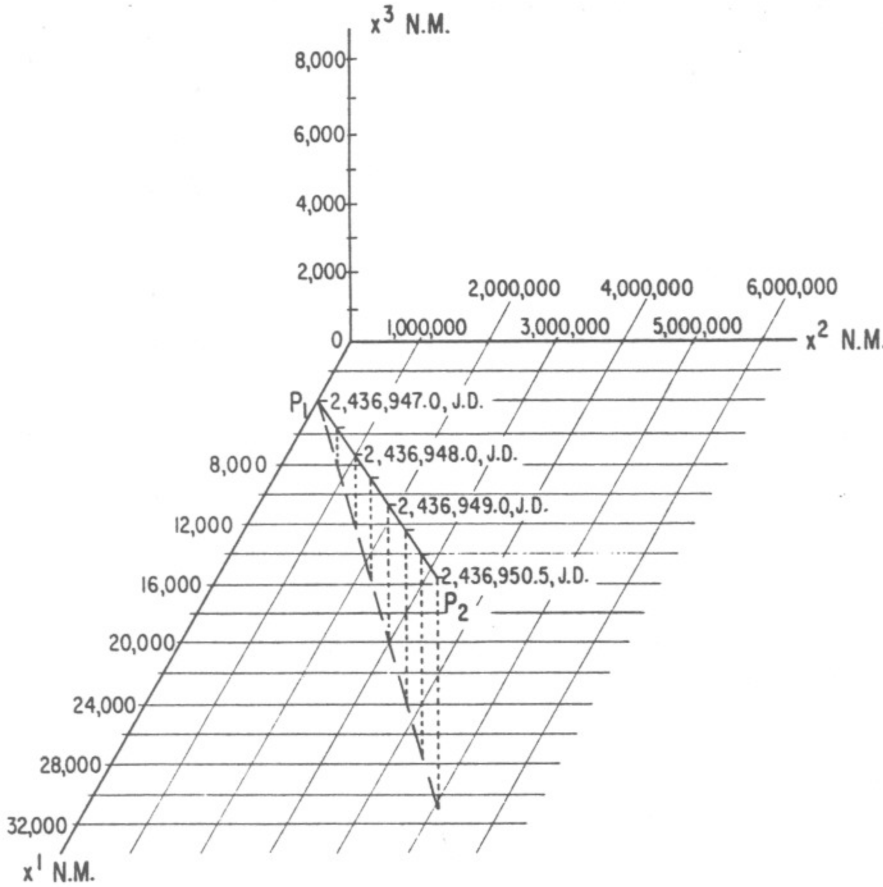

Fig. 2 "Initial Solution" for a Specific Time Interval

above paragraph could be. In Fig. 2, the motion from a point P_1 to another point P_2 is shown as being uniform, for given Julian dates. The forces per unit mass exerted upon this particular motion - the simplest that can be assumed - by different planets are given in g's in Fig. 3. From this graph an upper limit for the effect of neglecting, say, Jupiter is found to be about ten million feet in positional error for 75 hrs of flight time, a magnitude considerably larger than the numerical error of our solutions (see Part 3). It should, however, be noted that just the addition of another term in Eq. (3) would not avoid this error; to account for such effect, the summations in Eq. (10) must be adequately arranged.

By use of ephemeris tables, the coordinates of the Moon and the Earth, in our Galilean system, $0 - x^1, x^2, x^3$ at specific intervals of time, can be easily obtained. Although these data are given at discrete points, we shall write

$$X_S^i = d^i = \text{constant}$$

$$X_E^i = x_E^i(t) \qquad (i = 1, 2, 3) \tag{2}$$

$$X_M^i = x_M^i(t)$$

and suppose these functions to be - as they are - continuous in the interval of time in which the motion takes place. We shall see that, with our formulation of the problem and method of solution, the data can be treated as given by functional relations without direct reduction to analytic expressions.

For this physical model, the equations of motion of a particle of negligible mass - with respect to the gravitational masses involved - take in our system of coordinates the following simple form.

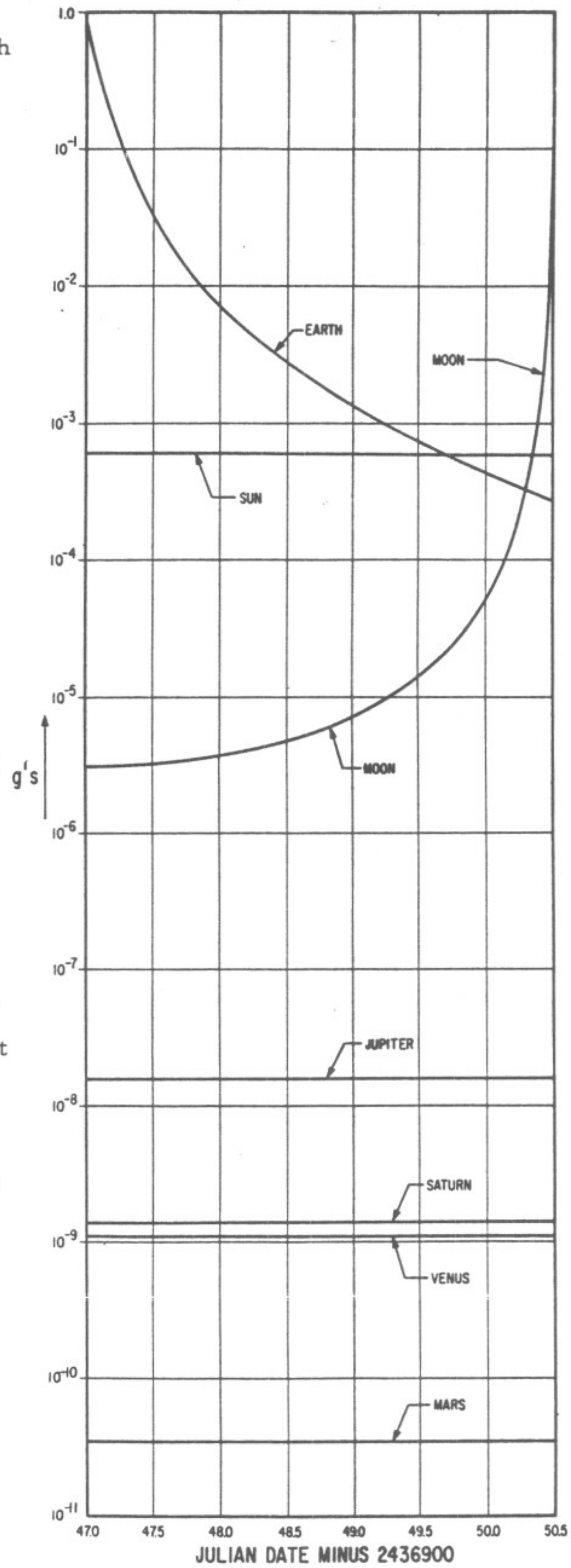

Fig. 3 Influence in g's of Different Planets Along the "Initial Solution"

$$\ddot{x}^i = K_S\left[d^i - x^i\right]\left\{\sum_{j=1}^{3}\left[d^j - x^j\right]^2\right\}^{-\frac{3}{2}}$$

$$+ K_E\left[x_E^i(t) - x^i\right]\left\{\sum_{j=1}^{3}\left[x_E^j(t) - x^j\right]^2\right\}^{-\frac{3}{2}} \tag{3}$$

$$+ K_M\left[x_M^i(t) - x^i\right]\left\{\sum_{j=1}^{3}\left[x_M^j(t) - x^j\right]^2\right\}^{-\frac{3}{2}}, \quad (i, j = 1, 2, 3)$$

where K_S, K_E, and K_M are the products of the universal gravitational constant times the masses of the Sun, the Earth, and the Moon, respectively. For other interplanetary problems that require taking into account the presence of other, or of additional planets, the corresponding differential equations will remain of the same general form.

2. THE LIMIT PROBLEM

2.1 MATHEMATICAL FORMULATION

Equations (3) are of the form

$$\ddot{x}^i = g^i(x^j, t) \quad (i, j = 1, 2, 3) \tag{4}$$

and a solution is sought such that

$$x^i(t_1) = a^i \tag{5}$$

$$x^i(t_1 + T) = b^i$$

i.e. the solution that goes from P_1: (a^i) to P_2: (b^i) in the interval of time T. The constant t_1 is an additive constant and will be always taken equal to zero.

If a solution, $x^i = x^i(a^j, \alpha^j, t)$, of the initial value problem

$$x^i(0) = a^i$$

$$\dot{x}^i(0) = \alpha^i$$

were obtainable in closed form and the equations

$$x^i(a^j, \alpha^j, T) = b^i$$

for α^i had a solution, the limit problem defined above would be solved. This approach is only possible in trivial cases.

To use Picard's thoughts for the solution of our limit problem, let the $g^i(x^j, t)$ of Eq. (4) be the functions in the right hand side of Eq. (3) and

$$x^i(t) \equiv x^i(a^j, b^j, t) \qquad (6)$$

be a solution of (13) satisfying the conditions of Eq. (5). Then, the integral relation,

$$x^i(t) = a^i + \frac{t}{T}\left[(b^i - a^i) - \int_0^T (T - r)\, g^i\,(x^j(r), r)\, dr\right] + \int_0^t (t - r)\, g^i\,(x^j(r), r)\, dr, \qquad (7)$$

will be satisfied identically in t and, conversely, any set of functions $x^i(t)$ satisfying Eq. (7) is a solution of Eq. (3) plus Eq. (5), as can be verified by derivation.

The right-hand side of Eq. (7) may be interpreted in a dual role: As a vector operator, $\Gamma_i\left[x^j(t)\right]$, that permits obtaining sequentially from any given set of functions $x^i_h(t)$ another set,

$$x^i_{h+1}(t) = \Gamma_i\left[x^j_h(t)\right] \qquad (8)$$

that always verifies the conditions of Eq. (5), and as an operator that provides the means "to check" whether any given set of functions $\bar{x}^i(t)$ verifying Eq. (5) is a solution of Eq. (3). For, in that case,

$$\bar{x}^i(t) \equiv \Gamma_i\left[\bar{x}^j(t)\right]. \qquad (9)$$

With $g^i(x^j_h(t), t) \equiv \bar{g}^i_h(t)$, the operator Γ_i may be written in the compact form,

$$x^i_{h+1}(t) = a^i + \frac{t}{T}\left[(b^i - a^i) - \int_0^T (T - r)\,\bar{g}^i_h(r)\,dr\right] + \int_0^t (t - r)\,\bar{g}^i_h(r)\,dr \equiv \Gamma_i\left[x^j(t)\right]. \tag{10}$$

The operator $\dot{\Gamma}_i$ *, giving the derivatives of $x^i_{h+1}(t)$ from $x^i_h(t)$, is defined by the similar formula,

$$\dot{x}^i_{h+1}(t) = \frac{1}{T}\left[(b^i - a^i) - \int_0^T (T - r)\,\bar{g}^i_h(r)\,dr\right] + \int_0^t \bar{g}^i_h(r)\,dr \equiv \dot{\Gamma}_i\left[x^i_h(t)\right]. \tag{11}$$

Both Γ_i and $\dot{\Gamma}_i$ may be obtained from Eq. (4) by formal integrations.

2.2 METHOD OF SOLUTION

Picard's iterative method - with some essential modifications - is used to construct solutions.

Take, as "initial guess" of the solution, an arbitrary set of functions $x^i_0(t)$, verifying the conditions of Eq. (5). Repeated use of the operator Γ_i, and of $\dot{\Gamma}_i$ yields the double sequence of functions

$$\Gamma_i\left[x^j_0(t)\right],\ \Gamma^2_i\left[x^j_0(t)\right],\ \text{-----}\ \Gamma^h_i\left[x^j_0(t)\right] = x^i_h(t) \tag{12}$$

$$\dot{\Gamma}_i\left[x^j_0(t)\right],\ \dot{\Gamma}_i\left[x^j_1(t)\right],\ \text{-----}\ \dot{\Gamma}_i\left[x^j_h(t)\right] \tag{13}$$

If the limit

$$\lim_{h\to\infty} \Gamma^h_i\left[x^j_0(t)\right] = \lim_{h\to\infty} x^i_h(t) = x^i(t) \tag{14}$$

exists, then

$$\dot{\Gamma}_i\left[\lim_{h\to\infty} x^j_h(t)\right] = \dot{x}^i(t) \tag{15}$$

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*Although $\dot{\Gamma}_i\left[x^j(t)\right]$ is called an operator, it is nothing but a convenient notation; for $\dot{\Gamma}^h_i\left[x^j_0(t)\right]$ is not well defined.

and $x^i(t)$ and $\dot{x}^i(t)$ are the coordinates and the velocity components of our original dynamical problem.

To obtain the conditions for convergence for a single differential equation, Picard wrote the sequences (12) and (13) in the form of series,

$$x_0^i(t) + \left[x_1^i(t) - x_0^i(t)\right] + \text{-----} \left[x_h^i(t) - x_{h-1}^i(t)\right] + \text{-----}$$

$$\dot{x}_0^i(t) + \left[\dot{x}_1^i(t) - \dot{x}_0^i(t)\right] + \text{-----} \left[\dot{x}_h^i(t) - \dot{x}_{h-1}^i(t)\right] + \text{-----}$$

and found that, if the right hand side of Eq. (3) (i= 1, only) verifies Lipschitz's condition and both T and the absolute value of the slope of the straight line joining the points P_1 and P_2 (in the x-t plane) are sufficiently small, a solution of Eq. (4) plus Eq. (5) exists and is unique. Instead of discussing these questions in general terms, something which always implies the "a priori" selection of dominant constants, we shall give numerical techniques that provide simultaneously the solution* and the analysis of the specific problem at hand.

In essence, these numerical techniques consist of the following steps:

i. - Divide the time interval (0, T) into n parts, which need not be equal.

ii. - Compute the values of $x_E^i(t)$ and $x_M^i(t)$ at those n values of t. (If, by electronic machines, these values are computed as needed, considerable storage space is saved, but the computational time is increased.)

iii. - Select an arbitrary set of functions $x_0^i(t)$ that verify conditions of Eq. (5)

iv. - Set an algorithm to carry out the operations in Eqs. (10) and (11); i. e. arrange the necessary calculations to obtain

*In all the cases studied solutions were obtained; these included trajectories for T = 84 hours.

$\Gamma_i\left[x_h^j(t)\right]$ and $\dot{\Gamma}_i\left[x_h^i(t)\right]$ from $x_h^i(t)$ at the selected values of t.

v. - Examine whether the differences

$$\left|\Gamma_i\left[x_h^j(t)\right] - x_h^i(t)\right| \tag{16}$$

$$\left|\dot{\Gamma}_i\left[x_h^j(t)\right] - \dot{x}_h^i(t)\right| \tag{17}$$

are, for all the n values of t, smaller than prescribed numbers - defining the accuracy of the iteration process - from a certain h onward.

vi. - The examination of numerical results shows that each of the sequences (12) and (13) separates into two distinct sequences,** corresponding to the odd and even iterations for each of the $x^i(t)$,

$$\begin{array}{l} x_1^i(t),\ x_3^i(t),\ \text{-----}\ x_{2h+1}^i(t)\ \text{-----} \\ x_2^i(t),\ x_4^i(t)\ \ \text{-----}\ x_{2h}^i(t)\ \text{-----} \end{array} \tag{18}$$

(and the same for their derivatives).

One observes further that these double sequences converge to a common limit, or diverge from each other, depending - as it should be - on the physical nature of the problem under scrutiny. In both cases, rates of convergence, or of divergence, of the sequences $x_{2h+1}^i(t)$ and $x_{2h}^i(t)$ are extremely small (very near unity) and almost identical. Therefore, the following simple, but essential, modification in the iteration process was introduced: After an arbitrarily selected number m of iterations, an average is taken of the last two, and the result,

$$x_{0,1}^i(t) \equiv \frac{x_m^i(t) + x_{m-1}^i(t)}{2}, \tag{19}$$

is used as a new initial guess, $x_{0,1}^i(t)$ of the solution. This mixed cycle of m-iterations and an average is repeatedly applied, and the resulting sequence for the $x_{0,k}^i(t)$ can be described by use of the operator Γ_i in

**The mathematical reasons for this phenomenon will be discussed in a forthcoming paper.

the following manner,

$$x^i_{0,1}(t) = \frac{\Gamma_i^{m-1}\left[x^j_0(t)\right] + \Gamma_i^m\left[x^j_0(t)\right]}{2} \qquad (20)$$

$$x^i_{0,2}(t) = \frac{\Gamma_i^{m-1}\left[x^j_{0,1}(t)\right] + \Gamma_i^m\left[x^j_{0,1}(t)\right]}{2}$$

and so forth. To the sequence

$$\Gamma_i\left[x^j_{0,1}(t)\right],\ \Gamma_i\left[x^j_{0,2}(t)\right], \text{-----},\ \Gamma_i\left[x^j_{0,k}(t)\right] \text{-----} \qquad (21)$$

there corresponds the sequence,

$$\dot{\Gamma}_i\left[x^j_{0,1}(t)\right],\ \dot{\Gamma}_i\left[x^j_{0,2}(t)\right],\ \text{-----},\ \dot{\Gamma}_i\left[x^j_{0,k}(t)\right]\text{-----} \qquad (22)$$

for the $\dot{x}^i(t)$.

The examination of results is not, however, done on the sequences (20) and (21) directly. Using the role of Γ_i indicated by (9), the differences

$$\left|\Gamma_i\left[x^j_{0,k}(t)\right] - x^j_{0,k}(t)\right| \qquad (23)$$

are examined to determine whether $x^i_{0,k}(t)$ is a solution of Eq. (4) and (5)*. It is of importance to examine also the double sequences,

$$\Gamma_i\left[x^j_{0,k}(t)\right],\ \Gamma_i^3\left[x^j_{0,k}(t)\right],\ \text{-----},\ \Gamma_i^{m-1}\left[x^j_{0,k}(t)\right] \qquad (24)$$

$$\Gamma_i^2\left[x^j_{0,k}(t)\right],\ \Gamma_i^4\left[x^j_{0,k}(t)\right],\ \text{-----},\ \Gamma_i^m\left[x^j_{0,k}(t)\right] \qquad (25)$$

(written for m even) and those corresponding to $\dot{x}^i(t)$, for the nature of the specific problem at hand is defined by the convergence, or divergence, of these sequences.

In general, a solution is attained after 5, 6, or 10 complete cycles - m equals 20 - even when the two original sequences (18) are divergent. These are truly remarkable numerical events, for a simple modification to Picard's method - the averaging process - permits construction of

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*In this case, $\dot{\Gamma}_i\left[x^j_{0,k}(t)\right] = \frac{d}{dt}\Gamma_i\left[x^j_{0,k}(t)\right]$

solutions in cases where the results of Picard's straight iteration process tell us that the solution is not necessarily unique.

2.3 FACTORS AFFECTING THE ACCURACY AND THE RAPIDITY OF THE PROCESS

These factors are:

i. - The number and distribution of the n values of t, between O and T

ii. - The formula used for the numerical integrations in Γ_i and $\dot{\Gamma}_i$

iii. - The number of digits carried out in the computations

iv. - The initial guess of the solution

v. - The number m defining the cycle described in 2.2vi

These factors are related to each other and to the form in which astronomical data are available. If the n time-values are not coincident with those for which $x_E^i(t)$ and $x_M^i(t)$ are given, an interpolation formula will have to be used to acquire the needed values of x_E^i and x_M^i. This formula should fit consistently with the integration scheme for the calculation of Γ_i and $\dot{\Gamma}_i$; i.e., neither the interpolation nor the integration formula should imply an order of approximation which the other makes it impossible to reach. In this connection, some astronomical work comes into the picture, for the interpolation formula should be one of those used in astronomy.

With these ideas in mind, since computing machines have a limited storage capacity and to operate them is costly, the worker must make a compromise as to what computational arrangement is most adequate for his purpose.

Once these choices are made, and assuming that our process of constructing solutions is converging, the question arises of whether for different n's there will result different solutions; the optimum solution to the physical problem is given by that maximum division of the interval

(O, T) into $\bar{n}$ parts beyond which the accuracy of the solution cannot be improved. This, in turn, is related to the number of digits carried in the computations: The value of any term in the sums representing the integrals in Γ_i and $\dot{\Gamma}_i$ for $n = \bar{n}$ must always be such that, its own individual error is adequately bound, and large enough to alter at least the last significant digit of all possible partial sums. For any subdivision beyond such $\bar{n}$, the integration process will degenerate; this should be first detected by fluctuations in the values of $x^i(t)$ in some part of the trajectory.

The number of digits carried in the computations plays another, very important role.

The sequence of powers of the operator Γ_i applied to $x_0^i(t)$ represents successive corrections to the initial guess. Thence, it is clear that the number of digits carried in the computations determines the degree of accuracy - and, therefore, of speed - with which these corrections are performed, i.e. the number h for which

$$\left|\Gamma_i^{h+1}\left[x_0^j(t)\right] - \Gamma_i^h\left[x_0^j(t)\right]\right| < \epsilon$$

is a function of the number of digits carried in the arithmetic operations.

From the above, intuitive interpretation of the sequences Γ_i^h, it is also clear that the better the initial guess is, the faster will be the convergence to the solution. Since for the cases where the straight iteration process is not convergent it is not possible to guarantee the uniqueness of the solution, one should ask whether different choices of $x_0^i(t)$ would yield different, or differently converging solutions.

The same question is applicable to the choice of m: Does the repeated application of our complete cycle yield different solutions for different m's? The dependence on m of the value of k for which

$$\left|\Gamma_i^2\left[x_{0,k}^j(t)\right] - \Gamma_i\left[x_{0,k}^j(t)\right]\right| < \epsilon$$

from k on, is obvious.

3. NUMERICAL RESULTS

3.1 SIMPLIFIED PHYSICAL MODEL

Inasmuch as in the present work we are not concerned with the analysis of a specific mission, assumptions have been made that simplify considerably the computational work but leave unaltered the questions pertaining to the method.

These simplifications are:

i. - The Earth moves about the Sun in a perfect circle.

ii. - The Moon also moves about the Earth in a perfect circle.

iii. - The integrals in Γ_i and $\dot{\Gamma}_i$ are calculated by Simpson's rule.

To obtain the functions $x^i_E(t)$ and $x^i_M(t)$, to define the problem, and to present the results, the following systems of coordinates are used:

The Galilean system with center at the Sun, S - ζ^1, ζ^2, ζ^3

The fundamental system of reference, 0 - x^1, x^2, x^3

A system of coordinates with center at the Earth, E- ξ^1, ξ^2, ξ^3, and always parallel to 0 - x^1, x^2, x^3.

A system of coordinates with center at the Moon, M- η^1, η^2, η^3 and always parallel to 0-x^1, x^2, x^3

All these systems are well defined by Fig. 1. The data required in Eq. (3) can be written immediately.

For the Sun's coordinates,

$$d^1 = -R$$

$$d^2 = 0$$

$$d^3 = 0,$$

R = 80728492 N. M.

For the Earth's coordinates,

$$x_E^1(t) = R\left[\cos(\dot{\Omega} t) - 1\right]$$

$$x_E^2(t) = R \sin(\dot{\Omega} t)$$

$$x_E^3(t) = 0,$$

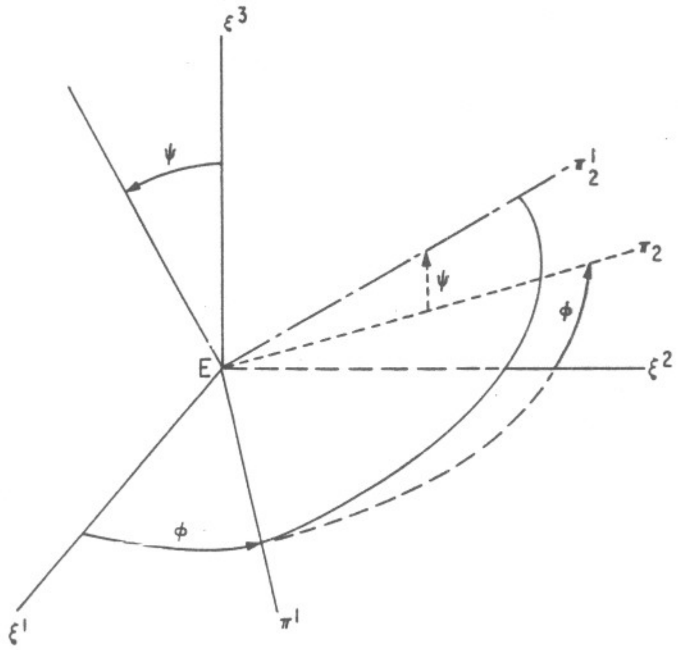
$$\dot{\Omega} = 7.1676658 \times 10^{-4} \text{ rad./hr.}$$

For the Moon's coordinates

$$x_M^1(t) = x_E^1(t) + r\left[\cos\phi \cos(\omega_0 + \dot{\omega} t) - \sin\phi \cos\psi \sin(\omega_0 + \dot{\omega} t)\right]$$

$$x_M^2(t) = x_E^2(t) + r\left[\sin\phi \cos(\omega_0 + \dot{\omega} t) + \cos\phi \cos\psi \sin(\omega_0 + \dot{\omega} t)\right]$$

$$x_M^3(t) = r \sin\psi \sin(\omega_0 + \dot{\omega} t)$$



(For December 1961, the Moon's center moves roughly on this plane.)

Fig. 4 Location of Plane of the Idealized Moon's Motion with r = 207561.40 N. M., $\dot{\omega}$ = 9.5823497 x 10^{-3} rad./hr., ϕ = 2.4434609 rad., ψ = 4.0944 x 10^{-2} rad, and ω_0 a constant value which is specified later on. Thence, the Moon moves in the plane E - π_1, π_2', defined in the system E - ξ^1, ξ^2, ξ^3 as follows: π_1 and π_2 are the positions of ξ^1 and ξ^2 after a ϕ - rotation about ξ^3, and π_2' is the position of π_2 after a ψ - rotation about π_1. (See Fig. 4)

3.2 DESCRIPTION AND SCOPE OF THE NUMERICAL EXAMPLES

The principal aim of the numerical work was to test, or to try Picard's iterative method. A systematic array of computations was prepared for the simplified model described in the preceding section.

The schedule for the set of runs, labeled Run I-1 is given below.

Runs I : T = 74 hrs.; $\overline{P_2M}$ = 1827 N. M.* at t = T

Run I-1 : $\overline{P_1E}$ = 10500 N. M. at t = 0.

Run I-2 : $\overline{P_1E}$ = 8000 N. M. " " " ".

Run I-3 : $\overline{P_1E}$ = 7000 N. M. " " " ".

Run I-4 : $\overline{P_1E}$ = 6000 N. M. " " " ".

Run I-5 : $\overline{P_1E}$ = 5000 N. M. " " " ".

Run I-6 : $\overline{P_1E}$ = 4000 N. M. " " " ".

Run I-7 : $\overline{P_1E}$ = 3700 N. M. " " " ".

These runs were started with an initial guess, $x_0^i(t)$, given by

$$(x_0^i - x_1^i)(t - T) - (x_0^i - x_2^i)\,t = 0 \qquad (26)$$

This motion is that shown in Fig. 2

---

* An input card was erroneously punched at the onset of the numerical work and was kept uncorrected all through the computation. $\overline{P_2M}$ was intended to be such that the final distance to the moon's surface would be 1000 N. M.

By direct use of Picard's iterative process, solutions were obtained after 400 iterations for Runs I-1, I-2, but not for Run I-3; convergence in this case would have required something of the order of 5×10^5 iterations. Furthermore, for these two runs the convergence was achieved only to 4 or 5 digits. A change to the cyclic process described in Section 2.2 was, therefore, indicated and the results presented in Tables I through VII were then obtained.

These trajectories represent a gradual approach to the Earth, carried as far as "practicable" (200 N. M.) from the surface of the actual Earth. It was intended to show, as it does, how the convergence process alters with the distance of P_1 to the "strong" singularity of the differential equations (3). The number of cycles required for convergency increases to 6 for $\overline{P_1 E}$ = 5000 N. M. with m constant. For this value of $\overline{P_1 E}$, the initial guess was changed from the arbitrary uniform motion of Eq. (26) to a more "educated" guess from the previous run; otherwise, the number of cycles required would have been considerably greater.

How the "odd" and "even" sequences begin to diverge from each other for $\overline{P_1 E}$ = 6000 N. M. is shown in Fig. 5, for two values of t, t = 7.4 hr. and t = 66.6 hr.

Finally, a trajectory, Run II, Table VIII, from 200 N. M. from the surface of the earth to impact on the moon was computed. Again the numerical results corroborate the marked influence of the "weak" singularity, located at the Moon. Ten cycles were required to achieve a convergency of the same order as that of Tables I through VII. Since this run is obviously the most interesting, a graphical presentation of the results is given in Figs. 6 through 10.

From Runs I-1 through I-7 the two curves of Figs. 11a and 11b were obtained. For a systematic analysis of an interplanetary mission, these curves are most important; they define the requirements for launching at a certain altitude and give the necessary information to know what will be

RUN I-4

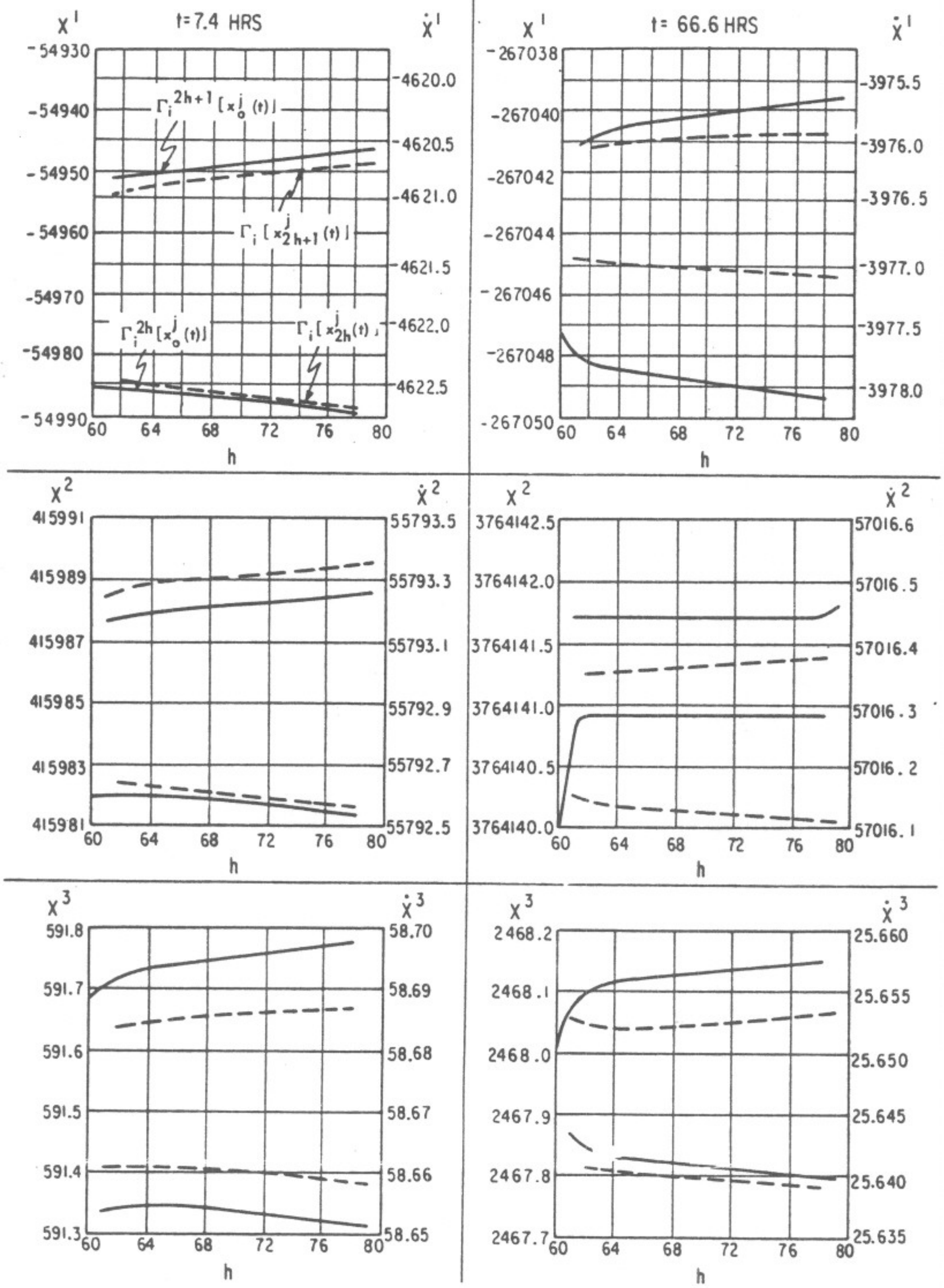


Fig. 5 SEQUENCES $\Gamma_i^{2h}[x_o^j(t)]$ and $\Gamma_i^{2h+1}[x_o^j(t)]$ and $\dot{\Gamma}_i[x_{2h}^j(t)]$ and $\dot{\Gamma}_i[x_{2h+1}^j(t)]$ vs h

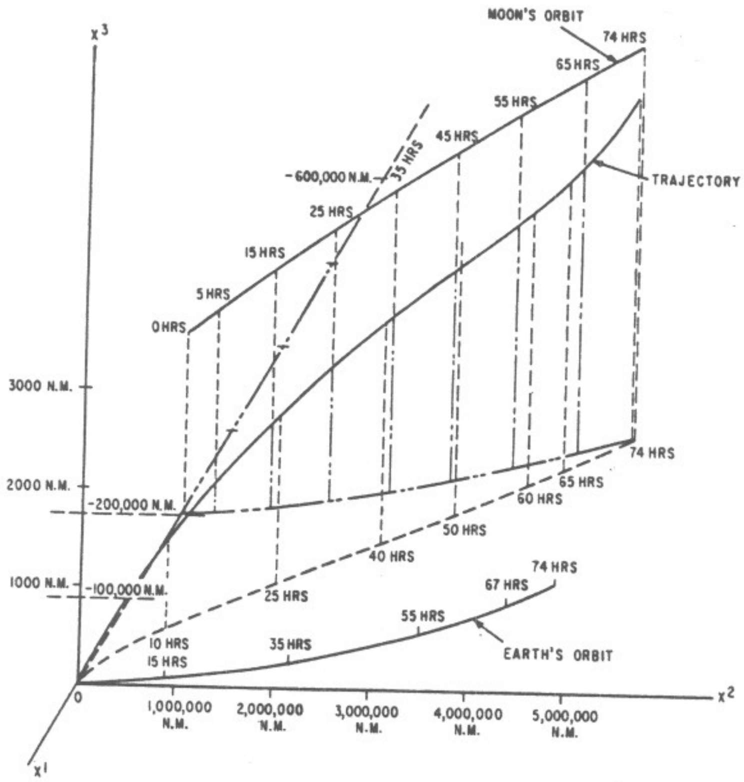


Fig. 6 Vehicle's, Earth's and Moon's Motions in the Galilean System of Reference

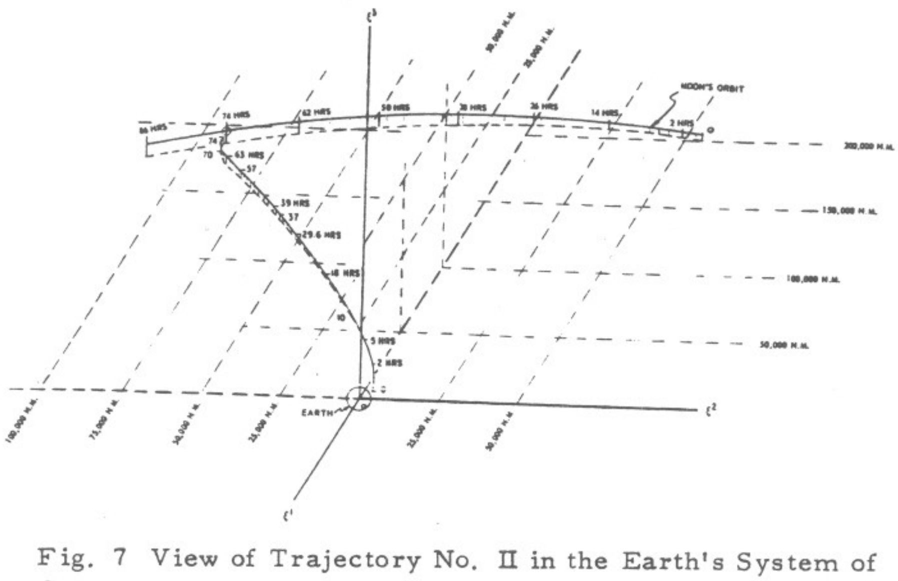


Fig. 7 View of Trajectory No. II in the Earth's System of Coordinates

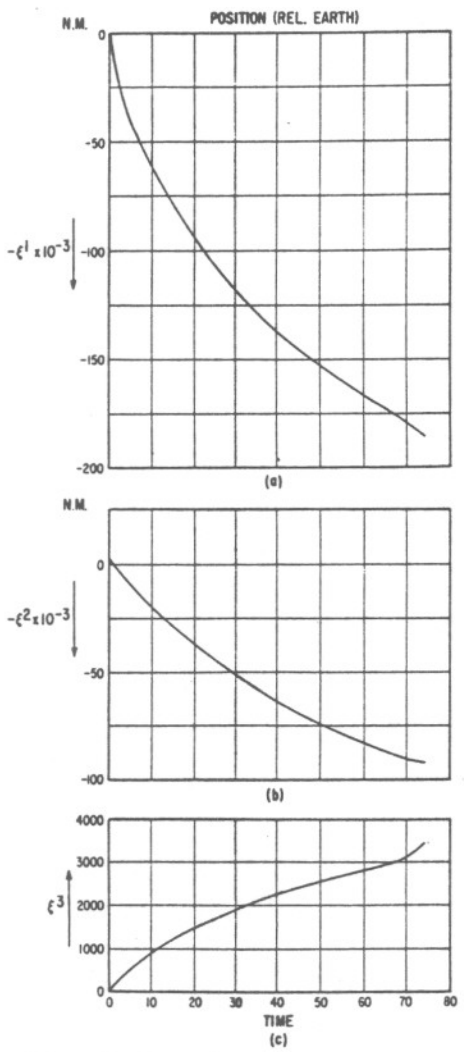


Fig. 8 Coordinates vs. Time, Trajectory II

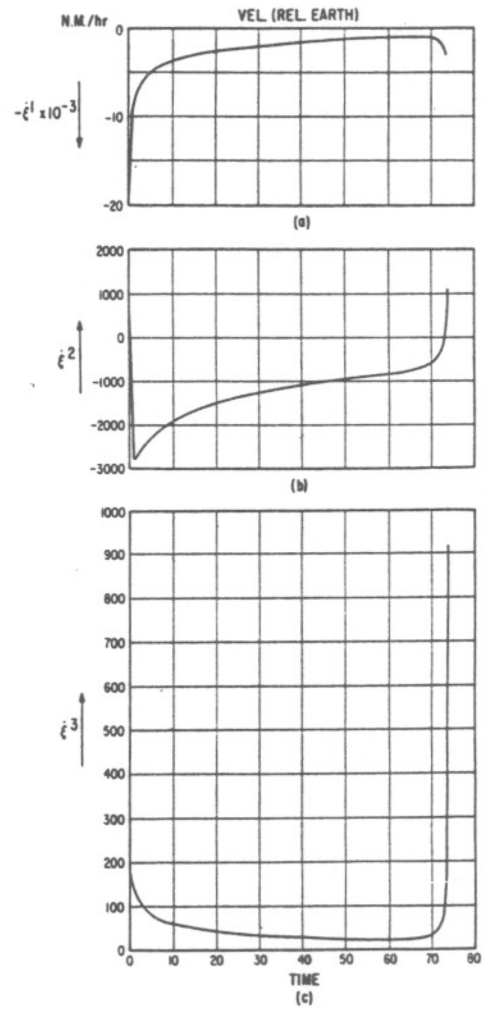


Fig. 9 Velocity Components vs Time, Trajectory II

the future history of the vehicle at P_2 and $t = T$, the vehicle will land on, circle, or leave the target planet, depending on the value of the velocity at P_2, $t = T$.

All these calculations were carried out with 12 digits. The values in the tables are typical. Naturally, the one thousand points calculated could not be presented.

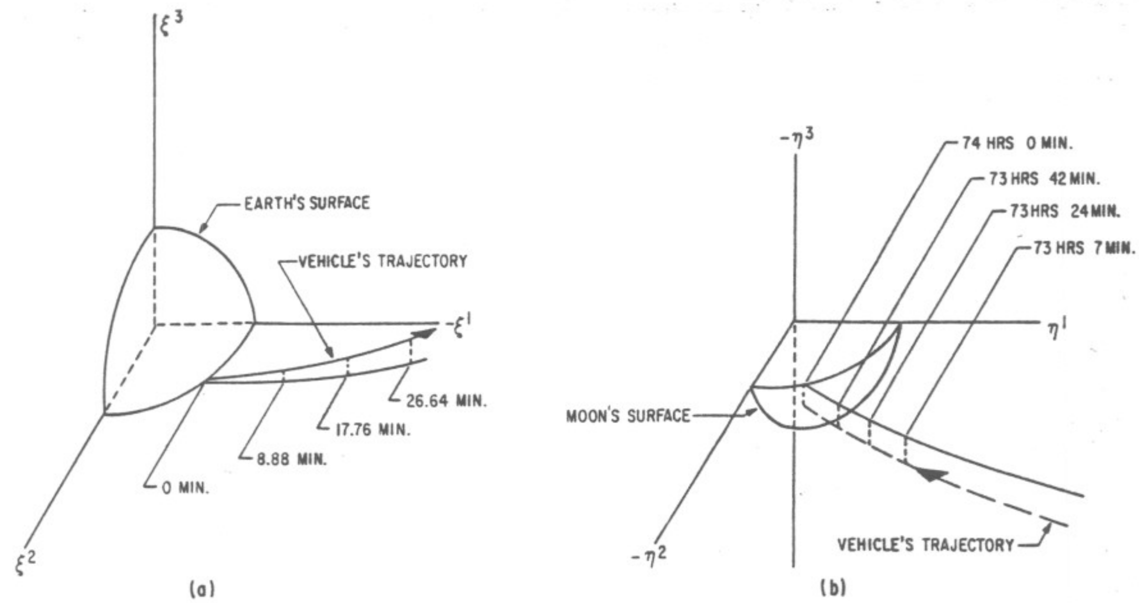


Fig. 10 Views of the Motion Near the Earth and Near the Moon

3.3 NUMERICAL ANALYSIS

The work of this section is an example of the numerical investigations that should be carried out to determine "the best" selection of n, m, and the initial guess. As noted previously, these inquiries are intimately connected with the formula used for the summations in Γ_i and $\dot{\Gamma}_i$, and to the number of digits carried in the computations. Therefore, the validity of these examples is limited to integrations by Simpson's rule, and to numerical operations with 12 digits.

The effect of the number of digits on the solution is shown in Table IX. The numbers printed are those to which the solution has converged after 80 iterations.

In Table X, the variation of the solution with n is clear. The printed numbers are those to which the solution has converged and it is evident that slightly different solutions result for different n's. Limitations in the memory capacity of the computer, for our special program, made it impossible to increase n beyond one thousand; therefore, it cannot be said that the "optimum" corresponds to the larger n tried. However, inspection of the numbers indicates that n = 1000 is very close to that optimum.

In Table XI, the change in the rate of convergence with m is shown, m = 6 being definitely the best of the two m's studied (20 and 6). An adequate selection of m may save a considerable amount of computational time.

Finally, in Table XII, the effect of the initial guess on the solution is shown. Two different initial guesses were tried, the uniform motion of Eq. (26) and a "guess" obtained from the previous run - that next and at a greater distance from the Earth - by a simple "stretching." The results show that the change in the "initial" guess affects only the rate of convergence to, but not the solution. Whether different solutions would would result for widely different "initial" guesses, cannot be ascertained.

Of course, all throughout these numerical investigations only one factor was changed at a time. The selection of initial conditions for testing the effect on the convergence process of varying m, n, the initial guess, and the number of digits was determined by purely logistic reasoning.

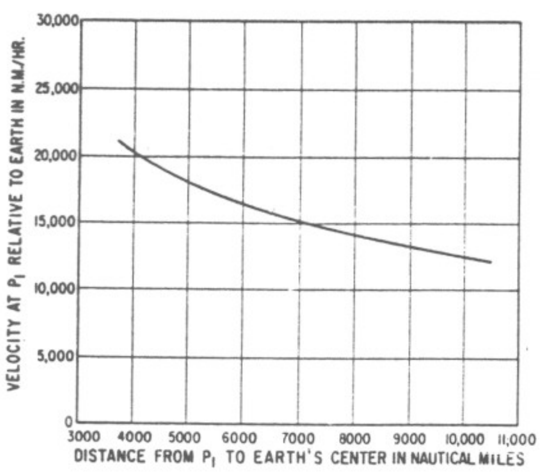


Fig. 11a - Initial Velocity vs Distance from Center of the Earth (P_2M constant, T=74 hrs.)

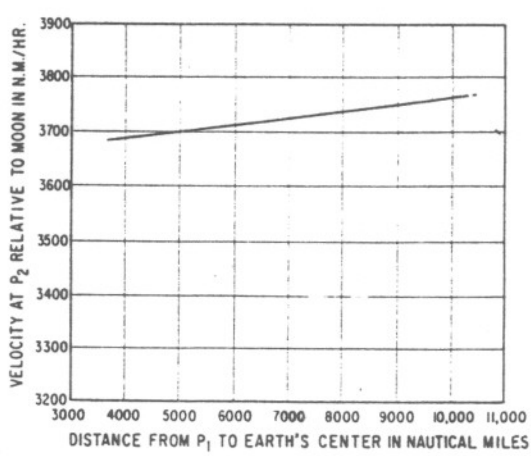


Fig. 11b - Final Velocity vs Initial Distance from Earth (P_2M constant, T=74 hrs.)

RUN I-1

TABLE I

P_1: (ξ = -8043, ξ^2 = 6749, ξ^3 = 0) at t = 0; P_2: (η^1 = 1037.82, η^2 = -1144.75, η^3 = -974.68) at t = T

$\overline{P_1E}$ = 10500, at t = 0; $\overline{P_2M}$ = 1827 at t = T. Lengths in N. M.

T = 74 hours n = 1001 m = 20 k = 5

x_o^i from $(x_o^i - x_1^i)(t - T) - (x_o^i - x_2^i)\, t = 0$. Input = $x^i_{o,k}(t)$, Output = $\Gamma_i\,[x^j_{o,k}(t)$.

t(hrs)		x^1 (N.M.)	x^2 (N.M.)	x^3 (N.M.)	$\dot{x}^1$ (N.M./hr)	$\dot{x}^2$ (N.M./hr)	$\dot{x}^3$ (N.M./hr)
0	INPUT	-8043.0000	6749.0000	0	-12180.889	58163.330	101.17823
	OUTPUT	-8043.0000	6749.0000	0	-12180.889	58163.330	101.17823
7.4	INPUT	-56352.105	421047.92	545.35016	-4632.8414	55858.609	57.463935
	OUTPUT	-56352.105	421047.92	545.35016	-4632.8414	55858.609	57.463950
14.8	INPUT	-86592.525	835300.81	917.06350	-3715.7929	56093.444	44.583728
	OUTPUT	-86592.525	835300.81	917.06351	-3715.7929	56093.444	44.583738
22.2	INPUT	-112728.36	1251090.1	1218.4498	-3400.4775	56273.905	37.404892
	OUTPUT	-112728.36	1251090.1	1218.4498	-3400.4775	56273.905	37.404897
29.6	INPUT	-137440.22	1668058.2	1476.1166	-3302.5710	56415.022	32.497369
	OUTPUT	-137440.22	1668058.2	1476.1166	-3302.5710	56415.022	32.497374
37.0	INPUT	-161874.11	2085970.7	1702.4135	-3314.9766	56530.916	28.825253
	OUTPUT	-161874.11	2085970.7	1702.4135	-3314.9766	56530.916	28.825263
44.4	INPUT	-186666.78	2504675.3	1904.7699	-3394.9517	56630.339	25.989683
	OUTPUT	-186666.78	2504675.3	1904.7699	-3394.9517	56630.339	25.989693
51.8	INPUT	-212238.78	2924075.3	2088.9079	-3523.7883	56719.896	23.916630
	OUTPUT	-212238.78	2924075.3	2088.9079	-3523.7883	56719.896	23.916640
59.2	INPUT	-238929.85	3344124.7	2261.5802	-3698.4268	56807.714	23.053020
	OUTPUT	-238929.85	3344124.7	2261.5802	-3698.4268	56807.714	23.053030
66.6	INPUT	-267154.74	3764877.4	2440.2742	-3950.7398	56917.831	26.947305
	OUTPUT	-267154.74	3764877.4	2440.2742	-3950.7398	56917.831	26.947315
74.0	INPUT	-298833.25	4187403.4	2999.9999	-5422.1597	57968.734	571.80040
	OUTPUT	-298833.25	4187403.4	2999.9999	-5422.1597	57968.734	571.79978

RUN I-2

TABLE II

P_1 : (ξ^1 = -6128, ξ^2 = 5142, ξ^3 = 0) at t = 0; P_2 : (η^1 = 1037.82, η^2 = - 1144.75, η^3 = -974.68) at t = T

$\overline{P_1E}$ = 8000, at t = 0; $\overline{P_2M}$ = 1827 at t = T. Lengths in N. M.

T = 74 hours n = 1001 m = 20 k = 5

x_o^i from $(x_o^i - x_1^i)(t - T) - (x_o^i - x_2^i)\, t = 0$ Input = $x^i_{o,k}(t)$, Output = $\Gamma_i\,[x^i_{o,k}(t)]$

t(hrs)		x^1 (N.M.)	x^2 (N.M.)	x^3 (N.M.)	$\dot{x}^1$ (N.M./hr)	$\dot{x}^2$ (N.M./hr)	$\dot{x}^3$ (N.M./hr)
0	INPUT	-6128.0000	5142.0000	0.	-14025.023	58416.614	115.05342
	OUTPUT	-6128.0000	5142.0000	0.	-14025.023	58416.614	115.05342
7.4	INPUT	-55605.452	418383.03	569.45450	-4631.1272	55823.074	58.150387
	OUTPUT	-55605.452	418383.03	569.45450	-4631.1271	55823.074	58.150382
14.8	INPUT	-85840.694	832606.95	943.59518	-3718.0885	56109.973	44.694204
	OUTPUT	-85840.694	832606.95	943.59519	-3718.0885	56109.973	44.694204
22.2	INPUT	-112017.13	1248593.0	1245.0647	-3408.8099	56308.069	37.342397
	OUTPUT	-112017.13	1248593.0	1245.0647	-3408.8099	56308.069	37.342397
29.6	INPUT	-136806.67	1665847.8	1501.9351	-3314.9355	56457.420	32.352519
	OUTPUT	-136806.67	1665847.8	1501.9351	-3314.9355	56457.420	32.352509
37.0	INPUT	-161342.21	2084091.8	1726.9583	-3329.8776	56577.716	28.628534
	OUTPUT	-161342.21	2084091.8	1726.9583	-3329.8776	56577.716	28.628534
44.4	INPUT	-186251.08	2503152.9	1927.6972	-3411.2943	56679.635	25.749204
	OUTPUT	-186251.08	2503152.9	1927.6972	-3411.2943	56679.635	25.749204
51.8	INPUT	-211946.46	2922923.4	2109.8748	-3540.6340	56770.620	23.623911
	OUTPUT	-211946.46	2922923.4	2109.8747	-3540.6341	56770.620	23.623911
59.2	INPUT	-238760.83	3343351.5	2280.0763	-3714.6837	56859.239	22.665166
	OUTPUT	-238760.83	3343351.5	2280.0763	-3714.6837	56859.239	22.665156
66.6	INPUT	-267097.94	3764487.4	2455.0451	-3964.2133	56969.870	26.254295
	OUTPUT	-267097.94	3764487.4	2455.0451	-3964.2133	56969.870	26.254295
74.0	INPUT	-298833.25	4187403.4	3000.0000	-5416.0586	58023.139	564.26813
	OUTPUT	-298833.25	4187403.4	3000.0000	-5416.0586	58023.139	564.26813

RUN I-3

TABLE III

P_1: (ξ^1 = -5362, ξ^2 = 4499.25, ξ^3 =0) at t = 0; P : (η^1 = 1037.82, η^2 = -1144.75, η^3 = -974.68) at t = T
$\overline{P_1 E}$ = 7000, at t = 0; $\overline{P_2 M}$ = 1827 at t = T. Lengths in N.M.
T = 74 hours n = 1001 m = 20 k = 5
x_0^i from $(x_0^i - x_1^i)(t - T) - (x_0^i - x_2^i)\, t = 0$. INPUT = $x_{0.k}^i(t)$, OUTPUT = $\Gamma_i\,[x_{0,k}^j(t)]$

t(hrs)		x^1(N.M.)	x^2(N.M.)	x^3(N.M.)	$\dot{x}^1$(N.M./hr)	$\dot{x}^2$(N.M./hr)	$\dot{x}^3$(N.M./hr)
0	INPUT	-5362.0000	4499.2500	0	-15027.142	58552.191	122.58228
	OUTPUT	-5362.0000	4499.2500	0	-15027.142	58552.191	122.58228
7.407	INPUT	-55328.336	417634.10	580.55973	-4625.7715	55808.623	58.393653
	OUTPUT	-55328.336	417634.10	580.55973	-4625.7715	55808.623	58.393653
14.815	INPUT	-85571.120	832266.95	955.83139	-3717.4297	56117.977	44.710857
	OUTPUT	-85571.120	832266.95	955.83139	-3717.4297	56117.977	44.710857
22.222	INPUT	-111783.06	1248760.5	1257.4364	-3411.7138	56323.587	37.290831
	OUTPUT	-111783.06	1248760.5	1257.4364	-3411.7138	56323.587	37.290831
29.630	INPUT	-136628.05	1666562.3	1514.0494	-3320.1429	56476.447	32.268353
	OUTPUT	-136628.05	1666562.3	1514.0494	-3320.1429	56476.446	32.268353
37.037	INPUT	-161232.78	2085373.6	1738.5932	-3336.6209	56598.627	28.523632
	OUTPUT	-161232.78	2085373.6	1738.5932	-3336.6209	56598.627	28.523632
44.444	INPUT	-186220.60	2505013.4	1938.6925	-3419.0649	56701.626	25.627369
	OUTPUT	-186220.60	2505013.4	1938.6925	-3419.0648	56701.626	25.627369
51.852	INPUT	-212001.86	2925369.6	2120.0834	-3549.0464	56793.257	23.483411
	OUTPUT	-212001.86	2925369.6	2120.0834	-3549.0464	56793.257	23.483411
59.259	INPUT	-238906.71	3346387.9	2289.3217	-3723.4007	56882.325	22.497339
	OUTPUT	-238906.71	3346387.9	2289.3217	-3723.4007	56882.325	22.497338
66.667	INPUT	-267337.20	3768118.6	2463.0866	-3973.1046	56993.743	26.056053
	OUTPUT	-267337.20	3768118.6	2463.0866	-3973.1045	56993.742	26.056052
74.000	INPUT	-298833.25	4187403.4	2999.9999	-5413.5971	58046.483	561.04537
	OUTPUT	-298833.25	4187403.4	2999.9999	-5413.5968	58046.483	561.04527

RUN I-4

TABLE IV

P_1: (ξ^1 = -4596, ξ^2 = 3856.5, ξ^3 = 0) at t = 0; P : (η^1 = 1037.82, η^2 = -1144.75, η^3 = -974.68) at t = T
$\overline{P_1 E}$ = 6000, at t = 0; $\overline{P_2 M}$ = 1827 at t = T. Lengths in N.M.
T = 74 hours n = 1001 m = 20 k = 5
x_0^i from $(x_0^i - x_1^i)(t - T) - (x_0^i - x_2^i)\, t = 0$. INPUT = $x_{0,k}^i(t)$, OUTPUT = $\Gamma_i\,[x_{0,k}^j(t)]$

t(hrs)		x^1(N.M.)	x^2(N.M.)	x^3(N.M.)	$\dot{x}^1$(N.M./hr)	$\dot{x}^2$(N.M./hr)	$\dot{x}^3$(N.M./hr)
0.0	INPUT	-4596.0000	3856.5000	0	-16270.940	58721.743	131.90968
	OUTPUT	-4596.0000	3856.5000	0	-16271.012	58721.750	131.91022
7.4	INPUT	-54968.276	415984.97	591.54361	-4621.6904	55792.961	58.672862
	OUTPUT	-54968.278	415984.97	591.54364	-4621.6991	55792.957	58.672989
14.8	INPUT	-85170.069	830192.68	967.44899	-3718.4104	56125.668	44.760118
	OUTPUT	-85170.069	830192.68	967.44903	-3718.4117	56125.666	44.760159
22.2	INPUT	-111375.65	1246359.6	1268.8179	-3415.7904	56339.059	37.269355
	OUTPUT	-111375.65	1246359.5	1268.8179	-3415.7889	56339.059	37.269361
29.6	INPUT	-136232.86	1663873.0	1524.8751	-3325.9130	56495.551	32.212148
	OUTPUT	-136232.86	1663873.0	1524.8752	-3325.9101	56495.551	32.212136
37.0	INPUT	-160859.33	2082414.7	1748.6905	-3343.2665	56619.671	28.444716
	OUTPUT	-160859.33	2082414.7	1748.6905	-3343.2627	56619.672	28.444693
44.4	INPUT	-185872.83	2501795.0	1947.9322	-3426.0311	56723.759	25.528422
	OUTPUT	-185872.82	2501795.0	1947.9322	-3426.0268	56723.760	25.528392
51.8	INPUT	-211679.57	2921897.0	2128.3212	-3555.8533	56815.976	23.358602
	OUTPUT	-211679.57	2921897.0	2128.3212	-3555.8486	56815.977	23.358568
59.2	INPUT	-238605.46	3342663.4	2296.2998	-3729.4117	56905.258	22.318764
	OUTPUT	-238605.46	3342663.4	2296.2998	-3729.4068	56905.259	22.318723
66.6	INPUT	-267044.55	3764141.3	2467.9732	-3976.5213	57016.240	25.646098
	OUTPUT	-267044.55	3764141.3	2467.9733	-3976.5161	57016.241	25.646036
74.0	INPUT	-298833.26	4187403.4	2999.9999	-5411.1277	58071.026	557.63267
	OUTPUT	-298833.26	4187403.4	2999.9999	-5411.1205	58071.024	557.63162

$\dot{x}^3$ (N.M./hr)
122.58228
122.58228
58.393653
58.393653
44.710857
44.710857
37.290831
37.290831
32.268353
32.268353
28.523632
28.523632
25.627369
25.627369
23.483411
23.483411
22.497339
22.497338
26.056053
26.056052
561.04537
561.04527

$\dot{x}^3$ (N.M./hr)
131.90968
131.91022
58.672862
58.672989
44.760118
44.760159
37.269355
37.269361
32.212148
32.212136
28.444716
28.444693
25.528422
25.528392
23.358602
23.358568
22.318764
22.318723
25.646098
25.646036
557.63267
557.63162

TABLE V

RUN I-5

P_1: (ξ^1 = -3830, ξ^2 = 3213.75, ξ^3 = 0; P_2: (η^1 = 1037.82, η^2 = -1144.75, η^3 = -974.68) at t = T

$\overline{P_1E}$ = 5000, at t = 0; $\overline{P_2M}$ = 1827 at t = T. Lengths in N.M.

T = 74 hours n = 1001 m = 20 k = 6

$x_0^i(t)$ obtained by "stretching" solution I-4. INPUT = $x^i_{0,k}(t)$, OUTPUT = $\Gamma_i\,[x^j_{0,k}(t)]$

t(hrs)		x^1 (N.M.)	x^2 (N.M.)	x^3 (N.M.)	$\dot{x}^1$ (N.M./hr)	$\dot{x}^2$ (N.M./hr)	$\dot{x}^3$ (N.M./hr)
0	INPUT	-3830.0000	3213.7500	0	-17878.960	58948.287	143.87465
	OUTPUT	-3830.0000	3213.7500	0	-17878.977	58948.289	143.87478
7.407	INPUT	-54654.098	415068.97	604.30816	-4612.1998	55777.633	58.905854
	OUTPUT	-54654.100	415068.98	604.30819	-4612.2019	55777.632	58.905879
14.815	INPUT	-84849.252	829690.24	891.27299	-3717.1650	56135.209	44.766551
	OUTPUT	-84849.254	829690.24	891.27302	-3717.1652	56135.209	44.766579
22.222	INPUT	-111090.30	1246379.3	1282.6960	-3419.1074	56356.868	37.205978
	OUTPUT	-111090.30	1246379.3	1282.6961	-3419.1071	56356.868	37.205979
29.630	INPUT	-136007.87	1664458.5	1538.4011	-3331.9612	56517.227	32.114394
	OUTPUT	-136007.87	1664458.5	1538.4012	-3331.9606	56517.228	32.114391
37.037	INPUT	-160710.72	2083588.2	1761.6298	-3351.0650	56643.431	28.324726
	OUTPUT	-160710.73	2083588.2	1761.6298	-3351.0628	56643.431	28.324721
44.444	INPUT	-185811.53	2503569.1	1960.1128	-3434.9617	56748.711	25.389861
	OUTPUT	-185811.53	2503569.1	1960.1129	-3434.9608	56748.711	25.389855
51.852	INPUT	-211713.03	2924279.3	2139.5826	-3565.4633	56841.632	23.199300
	OUTPUT	-211713.03	2924279.3	2139.5827	-3565.4624	56841.633	23.199293
59.259	INPUT	-238738.33	3345658.7	2306.4433	-3739.2981	56931.380	22.127966
	OUTPUT	-238738.34	3345658.7	2306.4434	-3739.2971	56931.380	22.127957
66.667	INPUT	-267279.11	3767754.2	2476.6962	-3986.4200	57043.115	25.409632
	OUTPUT	-267279.11	3767754.2	2476.6963	-3986.4190	57043.115	25.408619
74.000	INPUT	-298833.26	4187403.4	2999.9998	-5408.7065	58097.143	554.01546
	OUTPUT	-296833.26	4187403.4	2999.9998	-5408.7049	58097.143	554.01521

TABLE VI

RUN I-6

P_1: (ξ^1 = -3064, ξ^2 = 2571.2, ξ^3 = 0) at t = 0, P_2: (η^1 = 1037.82, η^2 = -1144.75, η^3 = -974.68) at t = T

$\overline{P_1E}$ = 4000, at t = 0; $\overline{P_2M}$ = 1827 at t = T. Lengths in N.M.

T = 74 hours n = 1001 m = 20 k = 6

$x_0^i(t)$ obtained by "stretching" solution I-4. INPUT = $x^i_{0,k}(t)$, OUTPUT = $\Gamma_i\,[x^j_{0,k}(t)]$

t(hrs)		x^1 (N.M.)	x^2 (N.M.)	x^3 (N.M.)	$\dot{x}^1$ (N.M./hr)	$\dot{x}^2$ (N.M./hr)	$\dot{x}^3$ (N.M./hr)
0	INPUT	-3064.0000	2571.2000	0	-20083.544	59286.342	159.97450
	OUTPUT	-3064.0000	2571.2000	0	-20087.140	59286.690	160.00239
7.407	INPUT	-54269.601	413615.14	617.80850	-4601.8372	55761.124	59.158968
	OUTPUT	-54270.141	413615.03	617.81460	-4602.1304	55760.993	59.163343
14.815	INPUT	-84429.298	828234.43	995.58461	-3716.6632	56145.298	44.787818
	OUTPUT	-84429.913	828234.28	995.59208	-3716.6973	56145.261	44.789030
22.222	INPUT	-110686.14	1245035.6	1296.8556	-3423.3642	56375.804	37.153809
	OUTPUT	-110686.74	1245035.4	1296.8634	-3423.3001	56375.810	37.153767
29.630	INPUT	-135645.84	1663272.2	1552.0254	-3338.8378	56540.314	32.025281
	OUTPUT	-135646.39	1663272.0	1552.0329	-3338.7239	56540.344	32.024592
37.037	INPUT	-160405.77	2082582.0	1774.4992	-3359.4594	56668.752	28.211181
	OUTPUT	-160406.25	2082581.8	1774.5062	-3359.3167	56668.796	28.210118
44.444	INPUT	-185572.20	2502755.6	1972.0629	-3444.1899	56775.300	25.255168
	OUTPUT	-185572.60	2502755.5	1972.0693	-3444.0296	56775.353	25.253871
51.852	INPUT	-211543.49	2923665.6	2150.4462	-3574.9909	56868.934	23.039082
	OUTPUT	-211543.80	2923665.5	2150.4519	-3574.8191	56868.991	23.037614
59.259	INPUT	-238638.74	3345248.8	2315.9709	-3748.5428	56959.044	21.921144
	OUTPUT	-238638.96	3345248.8	2315.9757	-3748.3624	56959.101	21.919473
66.667	INPUT	-267243.87	3767549.8	2484.2668	-3994.2339	57070.907	25.049298
	OUTPUT	-267243.98	3767549.7	2484.2705	-3994.0438	57070.957	25.046954
74.000	INPUT	-298833.28	4187403.3	2999.9997	-5406.3556	58125.353	550.07819
	OUTPUT	-298833.28	4187403.3	2999.9997	-5406.1118	58125.322	550.04768

RUN I-7

TABLE VII

P_1: (ξ^1 = -2840, ξ^2 = 2380, ξ^3 = 0) at t = 0; P_2; (η^1 = 1037.82, η^2 = -1144.75, η^3 = -974.68) at t = T
$\overline{P_1E}$ = 3700, at t = 0; $\overline{P_2M}$ ≐ 1827 at t = T. Lengths in N.M.
T = 74 hours n = 1001 m = 6 k = 9
$x_0^i(t)$ obtained by "stretching" solution I-6. INPUT = $x_{0,k}^i(t)$, OUTPUT = $\Gamma_i\{x_{0,k}^j(t)\}$

t(hrs)		x^1(N.M.)	x^2(N.M.)	x^3(N.M.)	$\dot{x}^1$(N.M./hr)	$\dot{x}^2$(N.M./hr)	$\dot{x}^3$(N.M./hr)
0	INPUT	-2840.0000	2380.0000	0	-20910.594	59418.650	165.84133
	OUTPUT	-2840.0000	2380.0000	0	-20910.594	59418.650	165.84133
7.407	INPUT	-54146.154	413149.95	622.02038	-4598.1631	55756.013	59.232990
	OUTPUT	-54146.154	413149.95	622.02037	-4598.1631	55756.013	59.232990
14.815	INPUT	-84293.420	827769.31	1000.0331	-3716.4633	56148.571	44.793782
	OUTPUT	-84293.420	827769.31	1000.0331	-3716.4633	56148.571	44.793782
22.222	INPUT	-110555.27	1244606.5	1301.2564	-3424.7403	56381.872	37.137915
	OUTPUT	-110555.27	1244606.5	1301.2564	-3424.7403	56381.872	37.137915
29.630	INPUT	-135528.60	1662893.4	1556.2635	-3341.0647	56547.696	31.998239
	OUTPUT	-135528.60	1662893.4	1556.2635	-3341.0647	56547.696	31.998239
37.037	INPUT	-160307.01	2082260.8	1778.5081	-3362.1748	56676.843	28.176674
	OUTPUT	-160307.01	2082260.8	1778.5081	-3362.1748	56676.843	28.176674
44.444	INPUT	-185494.66	2502495.9	1975.7921	-3447.1718	56783.794	25.214130
	OUTPUT	-185494.66	2502495.9	1975.7921	-3447.1718	56783.794	25.214130
51.852	INPUT	-211488.49	2923469.8	2153.8439	-3578.0685	56877.654	22.990098
	OUTPUT	-211488.49	2923469.8	2153.8439	-3578.0685	56877.654	22.990098
59.259	INPUT	-238606.35	3345118.1	2318.9591	-3751.5310	56967.877	21.857579
	OUTPUT	-238606.35	3345118.1	2318.9591	-3751.5310	56967.877	21.857579
66.667	INPUT	-267232.30	3767484.6	2486.6506	-3996.7689	57079.773	24.937781
	OUTPUT	-267232.30	3767484.6	2486.6506	-3996.7689	57079.773	24.937781
74.000	INPUT	-298833.29	4187403.3	2999.9995	-5405.6538	58134.306	548.82671
	OUTPUT	-298833.29	4187403.3	2999.9995	-5405.6538	58134.306	548.82671

RUN II

TABLE VIII

P_1: (ξ^1 = -2840, ξ^2 = 2380, ξ^3 =0) at t = 0; P_2: (η^1 = 531.82596; η^2 = -589.74854, η^3 = -499.67821) at t = T

$\overline{P_1E}$ = 3700, at t = 0; $\overline{P_2M}$ = 940 at t = T. Lengths in N.M.

T = 74 hours n = 1001 m = 6 k = 10

x_0^i from $(x_0^i - x^i)(t - T) - (x_0^i - x_2^i)\,t = 0$, INPUT = $x_{0,k}^i(t)$, OUTPUT = $\Gamma_i\{x_{0,k}^i(t)\}$

t(hrs)		x^1(N.M.)	x^2(N.M.)	x^3(N.M.)	$\dot{x}^1$(N.M./hr)	$\dot{x}^2$(N.M./hr)	$\dot{x}^3$(N.M./hr)
0	INPUT	-2840.0000	2380.0000	0	-20908.927	59439.962	200.69884
	OUTPUT	-2840.0000	2380.0000	0	-20908.927	59439.962	200.69884
7.4	INPUT	-54138.279	412824.95	752.45050	-4603.1553	55763.857	71.736703
	OUTPUT	-54138.279	412824.95	752.45050	-4603.1553	55763.857	71.736703
14.8	INPUT	-84286.965	827079.05	1209.7826	-3720.1113	56154.359	54.242373
	OUTPUT	-84286.965	827079.05	1209.7826	-3720.1113	56154.359	54.242373
22.2	INPUT	-110545.94	1243538.0	1574.1153	-3427.4541	56386.604	44.952437
	OUTPUT	-110545.94	1243538.0	1574.1153	-3427.4541	56386.604	44.952437
29.6	INPUT	-135511.51	1661439.9	1882.3509	-3343.0244	56551.733	38.695650
	OUTPUT	-135511.51	1661439.9	1882.3509	-3343.0244	56551.733	38.695650
37.0	INPUT	-160277.20	2080414.8	2150.6238	-3363.4881	56680.363	34.009223
	OUTPUT	-160277.20	2080414.8	2150.6238	-3363.4881	56680.363	34.009223
44.4	INPUT	-185447.25	2500254.2	2388.0954	-3447.9241	56786.902	30.309783
	OUTPUT	-185447.25	2500254.2	2388.0954	-3447.9241	56786.902	30.309783
51.8	INPUT	-211418.88	2920828.8	2601.0686	-3578.3652	56880.414	27.374520
	OUTPUT	-211418.88	2920828.8	2601.0686	-3578.3652	56880.414	27.374520
59.2	INPUT	-238510.78	3342074.7	2795.4028	-3751.6145	56970.317	25.349829
	OUTPUT	-238510.78	3342074.7	2795.4028	-3751.6145	56970.317	25.349829
66.6	INPUT	-267110.87	3764035.7	2982.4325	-3997.9376	57081.885	26.203909
	OUTPUT	-267110.87	3764035.7	2982.4325	-3997.9376	57081.885	26.203909
74.0	INPUT	-299339.28	4187958.3	3474.9999	-6308.4840	58861.599	911.60593
	OUTPUT	-299339.28	4187958.3	3473.9999	-6308.4840	58861.599	911.60593

TABLE IX

P_1: (ξ^1 = -5362, ξ^2 = 4499.25 ξ^3 = 0) at t = 0; P_2: (η^1 = 1037.82, η^2 = -1144.75, η^3 = -974.68) at t = T
$\overline{P_1E}$ = 7000 at t = 0; $\overline{P_2M}$ = 1827 at t = T. Lengths in N.M.
T = 74 hours n = 1001 m = 20 k = 5 Number of Digits carried = 8, 12
x_0^i from $(x_0^i - x_1^i)(t - T) - (x_0^i - x_2^i)$ t = 0

Time	No. of Digits	x^1 (N.M.)	x^2 (N.M.)	x^3 (N.M.)	$\dot{x}^1$ (N.M./hr)	$\dot{x}^2$ (N.M./hr)	$\dot{x}^3$ (N.M./hr)
0	8	-5362.0000	4499.2500	0	-15027.	58551.	122.65
	12	-5362.0000	4499.2500	0	-15027.	58552.	122.58
7.407	8	-55328.	417631.	580.9	-4625.6	55808.3	58.42
	12	-55328.	417634.	580.55	-4625.7	55808.62	58.393
14.815	8	-85570.	832262.	956.4	-3717.3	56117.78	44.737
	12	-85570.	832266.	955.83	-3717.42	56117.97	44.7108
22.222	8	-11178*.	124854.	1258.1	-3411.6	56323.42	37.313
	12	-11178*.	1248760.	1257.43	-3411.71	56323.587	37.2908
29.630	8	-13662*.	1666554.	1514.9	-3320.1	56476.30	32.288
	12	-136628.	1666562.	1514.04	-3320.1	56476.44	32.2683
37.037	8	-16123*.	2085364.	1739.6	-3336.6	56598.49	28.541
	12	-161232.	2085373.	1738.59	-3336.6	56598.62	28.5236
44.444	8	-186219.	2505001.	1939.8	-3419.0	56701.50	25.642
	12	-186220.	2505013.	1938.69	-3419.0	56701.62	25.627
51.852	8	-212001.	2925355.	2121.3	-3549.0	56793.13	23.496
	12	-212001.	2925369.6	2120.08	-3549.0	56793.25	23.483
59.259	8	-238906.	3346371.	2290.6	-3723.4	56882.21	22.507
	12	-238906.	3346387.9	2289.32	-3723.4	56882.32	22.497
66.667	8	-267337.	3768099.	2464.5	-3973	56993.65	26.056
	12	-267337.	3768118.6	2463.08	-3973.1	56993.74	26.056
74.000	3	-298833.25	4187403.4	2999.9999	-5407.	58045.6	556.
	12	-298833.25	4187403.4	2999.9999	-5413.6	58046.48	561.04

TABLE X

P_1: (ξ^1 = -4596, ξ^2 = 3856.5, ξ^3 = 0) at t = 0; P_2: (η^1 = 1037.82, η^2 = -1144.75, η^3 = -974.68) at t = T
$\overline{P_1E}$ = 6000, at t = 0; $\overline{P_2M}$ = 1827 at t = T. Lengths in N.M.
T = 74 hours n = 1001, 751, 501, 251 m = 20 k = 5
x_0^i from $(x_0^i - x_1^i)(t - T) - (x_0^i - x_2^i)$ t = 0

Time	n	x^1 (N.M.)	x^2 (N.M.)	x^3 (N.M.)	$\dot{x}^1$ (N.M./hr)	$\dot{x}^2$ (N.M./hr)	$\dot{x}^3$ (N.M./hr)
0	1001	-4596.0000	3856.5000	0	-16271.	58721.7	131.91
	751	-4596.0000	3856.5000	0	-16294.	58751.1	131.84
	501	-4596.0000	3856.5000	0	-16360.	58833.3	131.66
	251	-4596.0000	3856.5000	0	-16693.	59241.0	130.82
7.4	1001	-54968.2	415984.9	591.543	-4621.6	55792.9	58.672
	751	-54965.2	415974.4	591.787	-4621.6	55792.8	58.690
	501	-54956.6	415945.2	592.459	-4621.6	55792.4	58.737
	251	-54913.2	415808.9	595.641	-4621.6	55790.7	58.968
14.81	1001	-85170.	830192.6	967.449	-3718.41	56125.66	44.7601
	751	-85166.	830182.1	967.793	-3718.42	56125.74	44.7716
	501	-85158.	830153.0	968.746	-3718.47	56125.95	44.8035
	251	-85113.	830016.4	973.285	-3718.78	56126.93	44.9581
37.0	1001	-160869.	2082414.	1748.69	-3343.26	56619.67	28.4446
	751	-160857.	2082407.	1749.21	-3343.33	56619.86	28.4508
	501	-160851.	2082387.	1750.68	-3343.52	56620.39	28.4679
	251	-160821.	2082295.	1757.69	-3344.45	56622.89	28.5505
59.2	1001	-238905.4	3342663.4	2296.29	-3729.40	56905.25	22.3187
	751	-238604.9	3342660.8	2296.92	-3729.48	56905.46	22.3114
	501	-238603.4	3342653.7	2298.67	-3729.68	56906.04	22.3289
	251	-238596.3	3342620.4	2307.04	-3730.68	56908.77	22.3658
66.6	1001	-267044.5	3764141.3	2467.97	-3976.5	57016.24	25.646
	751	-267044.5	3764140.3	2468.61	-3976.5	57016.45	25.645
	501	-267044.4	3764137.4	2470.38	-3976.77	57017.03	25.643
	251	-267044.4	3764124.2	2478.88	-3977.70	57019.79	25.635
74.0	1001	-298833.26	4187403.4	2999.9999	-5411.1	58071.02	557.63
	751	-298833.26	4187403.4	2999.9999	-5412.9	58073.93	560.44
	501	-298833.26	4187403.4	2999.9999	-5418.1	58082.10	568.35
	251	-298833.26	4187403.4	2999.9999	-5444.9	58123.45	608.22

TABLE XI

P_1: (ξ^1 = -4596, ξ^2 = 3856.5, ξ^3 = 0) at t = 0; P_2: (η^1 = 1037.82, η^2 = -1144.75, η^3 = -974.68) at t = T
$\overline{P_1E}$ = 6000 at t = 0; $\overline{P_2M}$ = 1827 at t = T. Lengths in N.M.
T = 74 hours n = 1001 m = 6, 20 k = 5
x_0^i from $(x_0^i - x_1^i)(t - T) - (x_0^i - x_2^i)\,t = 0$

Time	m	x^1(N.M.)	x^2(N.M.)	x^3(N.M.)	$\dot{x}^1$(N.M./hr)	$\dot{x}^2$(N.M./hr)	$\dot{x}^3$(N.M./hr)
0	20	-4596.0000	3856.5000	0	-162[illegible]1.	58721.7	131.910
	6	-4596.0000	3856.5000	0	-16270.940	58721.742	131.90968
7.4	20	-54968.27	415984.97	591.5436	-4621.69	55792.95	58.672
	6	-54968.278	415984.97	591.5436	-4621.6904	55792.960	58.67286
14.8	20	-85170.07	830192.68	967.4490	-3718.41	56125.66	44.7601
	6	-85170.071	830192.68	967.4490	-3718.4103	56125.668	44.76011
22.2	20	-111375.65	1246359.	1268.8179	-3415.78	56339.059	37.26936
	6	-111375.65	1246359.5	1268.817	-3415.7903	56339.059	37.269355
37.0	20	-160859.33	2082414.7	1748.6905	-3343.26	56619.67	28.4446
	6	-160859.33	2082414.7	1748.690	-3343.2665	56619.671	28.444716
51.8	20	-211679.57	2921897.0	2128.3212	-3555.84	56815.97	23.3585
	6	-211679.57	2921897.0	2128.321	-3555.8532	56815.976	23.358602
59.2	20	-238605.46	3342663.4	2296.2998	-3729.40	56905.25	22.3187
	6	-238605.46	3342663.4	2296.299	-3729.4117	56905.258	22.318763
66.6	20	-267044.55	3764141.3	2467.973	-3976.51	57016.24	25.6460
	6	-267044.55	3764141.3	2467.973	-3976.5212	57016.240	25.646098
74.0	20	-298833.26	4187403.4	2999.9999	-5411.12	58971.02	557.63
	6	-298833.26	4187403.4	2999.9999	-5411.127	58071.025	557.6325

TABLE XII

P_1: (ξ^1 = -3830, ξ^2 = 3213.75, ξ^3 = 0) at t = 0; P_2: (η^1 = 1037.82, η^2 = -1144.75, η^3 = -974.68) at t = T
$\overline{P_1E}$ = 5000, at t = 0; $\overline{P_2M}$ = 1827 at t = T. Lengths in N.M.
T = 74 hours n = 1001 m = 20 k = 6
$x_n^i(t)$ obtained from $(x_0^i - x_1^i)(t - T) - (x_0^i - x_2^i)\,t = 0$ and by "stretching" solution I-4.

Time	INITIAL GUESS	x^1(N.M.)	x^2(N.M.)	x^3(N.M.)	$\dot{x}^1$(N.M./hr)	$\dot{x}^2$(N.M./hr)	$\dot{x}^3$(N.M./hr)
0	"6000 N.M."	-3830.0000	3213.7500	0	-17878.9	58948.28	143.874
	"ST. LINE"	-3830.0000	3213.7500	0	-18306.274	58993.237	147.14138
7.407	"6000 N.M."	-54654.10	415068.9	604.3081	-4612.20	55777.63	58.9058
	"ST. LINE"	-54688.602	415062.37	604.64098	-4655.3335	55759.239	59.535976
14.815	"6000 N.M."	-84849.25	829690.24	981.2730	-3717.165	56135.209	44.76657
	"ST. LINE"	-84887.414	829680.92	981.65608	-3722.9121	56129.499	44.954047
22.222	"6000 N.M."	-111090.30	1246379.3	1282.696	-3419.107	56356.868	37.20597
	"ST. LINE"	-111126.85	1246369.5	1283.0700	-3410.6489	56357.218	37.216000
29.630	"6000 N.M."	-136007.87	166458.5	1538.401	-3331.960	56517.22	32.11439
	"ST. LINE"	-136040.77	1664449.2	1538.7431	-3316.3015	56520.985	32.032863
37.037	"6000 N.M."	-160710.7	2083588.2	1761.6298	-3351.06	56643.431	28.32472
	"ST. LINE"	-160739.00	2083580.0	1761.9296	-3331.2594	56649.231	28.190319
44.444	"6000 N.M."	-185811.53	2503569.1	1960.112	-3434.96	56748.711	25.38985
	"ST. LINE"	-185834.65	2503562.4	1960.3656	-3412.5989	56755.715	25.222352
51.852	"6000 N.M."	-211713.03	2924279.3	2139.582	-3565.46	56841.63	23.1992
	"ST. LINE"	-211730.69	2924274.2	2139.7862	-3541.4248	56849.229	23.007417
59.259	"6000 N.M."	-238738.3	3345658.7	2306.443	-3739.29	56931.380	22.1279
	"ST. LINE"	-238750.35	3345655.3	2306.5968	-3714.0100	56938.972	21.906243
66.667	"6000 N.M."	-267279.11	3767754.2	2476.696	-3986.41	57043.115	25.4086
	"ST. LINE"	-267285.32	3767752.5	2476.7983	-3959.6398	57049.616	25.085702
74.000	"6000 N.M."	-298833.26	4187403.4	2999.9999	-5408.70	58097.143	554.015
	"ST. LINE"	-298833.26	4187403.4	2999.9999	-5373.3810	58091.865	549.38024

REFERENCES

1. E. T. Benedikt, "Collision Trajectories in the Three-Body Problem", Journal of the Astronautical Sciences, Vol VI, No. 2, 1959, pp 17-24

2. G. D. Birkhoff, "Dynamical Systems", American Mathematical Society, Colloquium Publications, Vol IX., New York, 1927

3. E. Picard, "Sur l' application des méthodes d' approximations successives à l'étude de certaines équations differentielles ordinaires", Journal de Math. Pures et Appliquées, Vol. IX, 1893, pp 217-291

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