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Interplanetary Rocket Trajectories *

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I. Review of the Problems

This chapter is concerned with the mathematical problem of determining the optimal track along which a rocket vehicle may be transferred from a launching point on the surface of one planet of the solar system to a destination point on the surface of a second planet. The first planet will be designated the *departure planet* or D-planet and the second the *arrival planet* or A-planet.

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Optimization of the rocket trajectory has to be performed in respect to two major variables, (a) the propellant consumption, and (b) the time of transit. In general, it will be desirable to minimize both these factors by appropriate choice of the trajectory. However, these two requirements prove to be in direct conflict and a compromise will have to be sought. Until more powerful propellant combinations than are at present known become available, it will be necessary to weight this compromise very heavily in favor of a reduction in propellant consumption rather than in the time of transit. In the early years of planetary exploration, remotely controlled unmanned probes will presumably be employed and the time of transit will, in any case, be a factor of small importance. For such journeys, trajectories of least propellant expenditure will be followed. However, the saving in food, water, oxygen and the other essential supplies for a manned expedition following upon a reduction in the time of transit, may be greater than the consequent propellant increase. An overall economy will then have been effected. For such an expedition, a compromise trajectory will therefore have to be found.

Having made a decision with regard to the interplanetary trajectory to be followed, it will be necessary to consider the mode of entry into this trajectory from the launching point on the D-planet (usually the earth). It seems likely that the degree of control which will be exercised over the vehicle's motion during the ascent will be insufficiently precise to justify a direct transfer from the launcher into the interplanetary orbit. Instead, it is to be expected that the vehicle will first be transferred into a satellite orbit, as close as possible to a previously computed circular orbit, either in its final form or in sections to be assembled in space. Entry into the interplanetary orbit would then be made by the application of a carefully timed and directed impulse from the motors and initiated from the ground control station. In the case of a probe, this maneuver could only be performed successfully provided remote control of the orientation of the rocket axis was sufficiently precise to ensure that the thrust was properly directed. However, this is very largely a servo-engineering problem and will not be further considered in this Chapter. It seems feasible that suitably located compressed air jets or internal flywheels could be used to control the orientation of the rocket. Such devices would be linked to an element sensitive to the rocket's orientation and the combination formed into a closed servo control loop.

During its motion along the interplanetary trajectory between the two planets, it will be necessary to make sufficient observations of the rocket's position and velocity to establish the elements of its orbit. This, inevitably, will differ from the previously computed optimal trajectory and it will be necessary to apply a small correctional thrust as soon as possible after the divergence has been noticed. Again, a fine control over the rocket orientation will be necessary if this, and later, correctional thrusts are to be applied effectively. Also, a method of rapidly computing the magnitude and direction of a correctional impulse must be found.

A precise computation of the trajectory of the rocket as it moves between the planetary terminals would be a laborious process involving a step-by-step numerical integration best performed by a large scale automatic calculator. However, for many purposes, it will be sufficient to divide the transfer maneuver into three phases, (a) in proximity to the D-planet, (b) in interplanetary space and (c) in proximity to the A-planet. During each of these phases, the attraction of a single body is dominant and the influence of any other can be disregarded. Thus, during the first phase, by referring the motion to a frame moving with the D-planet, the effect of the sun's attraction can be reduced to that of a small perturbing force, due only to the non-uniformity of the sun's field in the neighborhood of the planet. It will be shown in Section XIII that this force causes spatial and velocity changes of magnitudes smaller than those to be expected as a result of the inevitable launching errors and consequently there appears to be little point in making any allowance for them. These errors can be corrected later on the trajectory, at the same time when the launching errors are corrected by the application of a small thrust from the motors. During the second phase, the vehicle will have receded to a great distance from the D-planet and its motion will be governed almost entirely by the attraction of the sun. The points which have been made in respect of the first phase apply equally to the third phase. It is clear, therefore, that the whole trajectory may be regarded as comprising a succession of Keplerian arcs and this approximation will be employed in much of the work described later.

It may happen that it is advantageous to introduce further phases into the space-ship's motion, during which it passes close to an attracting body other than one of the planets of departure and arrival. Thus, a rocket whose destination is Mars, might purposely be launched into a trajectory carrying it close to the moon with the object of acquiring momentum without the expenditure of propellant by making use of this body's attraction. Again, the trajectory might be chosen to permit a close encounter with one of the Martian moons, with the intention of conserving propellant at the destination planet. Such maneuvers are called *perturbation maneuvers*.

II. The Generalized Mayer Problem

As first pointed out by Cicala [1] and later amplified by Miele [2–4], the problem of minimizing the propellant expenditure of a rocket can be

arranged in the form of a calculus of variations problem of Mayer type and hence solved by known techniques [5]. In this Section, the Mayer problem will be stated in a generalized form, embracing within its scope all later problems concerning the calculation of optimal trajectories under various conditions. The solution to this problem will be stated and will then be employed as a basis for subsequent investigations.

The generalized Mayer problem may be stated thus: $x_i(t)$ $(i = 1, 2, \ldots, n)$ are n unknown functions of an independent variable t, which are to be defined for values of t extending over the range $t_0 \leqq t \leqq t_1$. These functions have to satisfy the m first order differential equations

$$\phi_j(t, t_0, t_1, \kappa_k, x_i, \dot{x}_i) = 0 \qquad (j = 1, 2, \ldots, m < n), \tag{1}$$

where the κ_k $(k = 1, 2, \ldots, q)$ are q parameters whose values are also unknown. The values taken by the functions $x_i(t)$ at the end points of the interval (t_0, t_1) will be denoted thus:

$$x_i(t_0) = x_{i0} \qquad x_i(t_1) = x_{i1}. \tag{2}$$

These values are to satisfy the p equations

$$\psi_l(t_0, t_1, x_{i0}, x_{i1}) = 0 \qquad (l = 1, 2, \ldots, p \leqq 2n + 2). \tag{3}$$

J is a given function of t_0, t_1, x_{i0}, x_{i1}, i.e.

$$J = J(t_0, t_1, x_{i0}, x_{i1}). \tag{4}$$

It is required to choose the functions x_i, subject to the *constraints* (1) and (3), the end points t_0, t_1, and the values of the κ_k, so that J is minimized.

To obtain a rigorous solution to this problem, it is necessary to state at the outset the assumptions which may be made concerning the analytical properties of the functions ϕ_j, etc., and to specify the set of functions x_i which are admissible for the purpose of minimizing J. However, apart from the statement that the x_i will always be supposed continuous and their first derivatives, the $\dot{x}_i$, continuous except at a finite number of points, these assumptions will not be made explicit, since it is our object to apply known results rather than to describe the most general circumstances in which they are valid. The reader interested in this aspect should consult Bliss [5].

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ denote Lagrange multipliers depending upon t and to be determined subsequently. Let F be the function defined by the equation

$$F = \lambda_j \phi_j. \tag{5}$$

Here, as elsewhere in this chapter, the usual summation convention is operative in respect to repeated lower case literal subscripts. Let $\nu_1, \nu_2, \ldots, \nu_p$ be constant multipliers, also to be determined later and let H be defined by the equation,

$$H = J + \nu_k \psi_k. \tag{6}$$

Then, the functions x_i which minimize J necessarily satisfy the second order differential equations

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) = 0 \qquad (i = 1, 2, \ldots, n), \tag{7}$$

at all points where the $\dot{x}_i$ are continuous. These are the *Euler equations*. Also, at the end points $t = t_0$, $t = t_1$, it is necessary that

$$\frac{\partial H}{\partial t_0} + \frac{\partial H}{\partial x_{i0}} \dot{x}_{i0} + \int_{t_0}^{t_1} \frac{\partial F}{\partial t_0} dt = 0, \tag{8}$$

$$\frac{\partial H}{\partial t_1} + \frac{\partial H}{\partial x_{i1}} \dot{x}_{i1} + \int_{t_0}^{t_1} \frac{\partial F}{\partial t_1} dt = 0, \tag{9}$$

$$\frac{\partial H}{\partial x_{i0}} - \left(\frac{\partial F}{\partial \dot{x}_i}\right)_0 = 0, \tag{10}$$

$$\frac{\partial H}{\partial x_{i1}} + \left(\frac{\partial F}{\partial \dot{x}_i}\right)_1 = 0, \tag{11}$$

$$\int_{t_0}^{t_1} \frac{\partial F}{\partial \kappa_k} dt = 0, \tag{12}$$

where $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, q$.

Together with the m constraints (1), the n Euler equations, which are of the second order, determine the $m + n$ functions x_i, λ_i, apart from $2(m + n)$ constants of integration. The $(2n + q + 2)$ Eqs. (8)–(12), together with the p constraints (3) and the $2m$ equations obtained by setting $t = t_0$, $t = t_1$ in Eqs. (1), determine the p constants ν_i, the end points t_0, t_1, the parameters κ_k and the $2(m + n)$ constants of integration.

The derivatives $\dot{x}_i$ may be discontinuous, but at each such discontinuity the *Weierstrass–Erdmann corner conditions* must be satisfied. These require that the following expressions shall be continuous at the discontinuity:

$$\frac{\partial F}{\partial \dot{x}_i}, \qquad F - \dot{x}_i \frac{\partial F}{\partial \dot{x}_i}.$$

Writing

$$K = F - \dot{x}_i \frac{\partial F}{\partial \dot{x}_i}. \tag{13}$$

it may be shown that

$$\frac{dK}{dt} = \lambda_j \frac{\partial \phi_j}{\partial t}. \tag{13a}$$

It follows that, if the ϕ_j are not explicitly dependent upon t,

$$K = \text{constant}. \tag{13b}$$

This is a first integral of the Euler equations. It is clear from the Weierstrass–Erdmann corner conditions that the constant right-hand member of this equation is not affected by a discontinuity.

III. Optimal Ascent into a Circular Orbit

To illustrate the conditions upon an optimal trajectory listed in Section II, the problem of transferring a rocket from a launching platform into a circular orbit about the earth with minimum expenditure of propellant will now be considered. It will be supposed that the vehicle is launched with zero initial velocity and that the initial direction of projection is not prescribed. The final velocity at the end of the powered trajectory is to be horizontal and of given magnitude U (satellite velocity) and at this terminal the rocket's height is to be Y above the launching point. It will be supposed that the propellant expenditure is a prescribed function of the time after launching t, so that the component of acceleration f due to the motor thrust is also a given function of t. It is required, therefore, to calculate the thrust direction program yielding the prescribed final conditions in the shortest possible time. For simplicity, aerodynamic forces, earth curvature and rotation and variation of gravity with height will be neglected. The effects of these factors can be expected to be small over the powered arc, which is relatively short.

Taking axes $0x$ horizontally, $0y$ vertically through the launching point 0, it will be assumed that the rocket's motion is confined to the plane $0xy$. Let (x, y) be the coordinates of the rocket at time t, (u, v) its velocity components in the directions of the axes, and θ the angle made by the direction of the motor thrust with the horizontal. Then the equations of motion of the rocket can be expressed in the form

$$\begin{aligned} \phi_1 &= \dot{x} - u = 0, \\ \phi_2 &= \dot{y} - v = 0, \\ \phi_3 &= \dot{u} - f\cos\theta = 0, \\ \phi_4 &= \dot{v} + g - f\sin\theta = 0, \end{aligned} \tag{14}$$

g being the gravitational acceleration. In the particular problem being considered, the constraints (1) take the form of Eqs. (14) and govern the five unknown functions x, y, u, v, θ (the x_i). We shall suppose that launching takes place at the instant $t = 0$. Then, the boundary conditions are

$$\begin{aligned} \psi_1 &= x_0 = 0, & \psi_6 &= y_1 - Y = 0, \\ \psi_2 &= y_0 = 0, & \psi_7 &= u_1 - U = 0, \\ \psi_3 &= u_0 = 0, & \psi_8 &= v_1 = 0. \\ \psi_4 &= v_0 = 0, \\ \psi_5 &= t_0 = 0, \end{aligned} \tag{15}$$

These represent the end constraints (3) for the problem. Finally,

$$J = t_1 \tag{16}$$

is to be minimized by choice of the functions x, y, u, v, θ satisfying Eqs. (14, 15).

The Lagrange function takes the form

$$F = \lambda_1(\dot{x} - u) + \lambda_2(\dot{y} - v) + \lambda_3(\dot{u} - f\cos\theta) + \lambda_4(\dot{v} + g - f\sin\theta) \tag{17}$$

and the Euler equations (7) are accordingly

$$\dot{\lambda}_1 = \dot{\lambda}_2 = \dot{\lambda}_3 + \lambda_1 = \dot{\lambda}_4 + \lambda_2 = 0, \tag{18}$$

$$f(\lambda_3\sin\theta - \lambda_4\cos\theta) = 0. \tag{19}$$

It follows from these equations that

$$\lambda_1 = -a, \qquad \lambda_2 = -c, \qquad \lambda_3 = at + b, \qquad \lambda_4 = ct + d, \tag{20}$$

$$\tan\theta = \frac{ct + d}{at + b}, \tag{21}$$

a, b, c, d being constants. We conclude, therefore, that the thrust direction program must be such that the tangent of the thrust angle θ is a bilinear function of the time. Since this result has been obtained without reference to the boundary conditions (15), it is implied that the thrust direction program will take this form for any mission to be accomplished in a uniform gravitational field and in the absence of aerodynamic forces. It is therefore of very great generality. Lawden [6–8] and Fried and Richardson [9] obtained this form of optimal thrust direction program when investigating other rocket maneuvers in a uniform field.

The function H [Eq. (6)] is now written down thus:

$$\begin{aligned} H = t_1 + \nu_1 x_0 + \nu_2 y_0 + \nu_3 u_0 + \nu_4 v_0 + \nu_5 t_0 \\ + \nu_6(y_1 - Y) + \nu_7(u_1 - U) + \nu_8 v_1. \end{aligned} \tag{22}$$

The conditions (8)–(12) then yield the equations

$$\begin{aligned} &\nu_5 + \nu_1\dot{x}_0 + \nu_2\dot{y}_0 + \nu_3\dot{u}_0 + \nu_4\dot{v}_0 = 0, \\ &1 + \nu_6\dot{y}_1 + \nu_7\dot{u}_1 + \nu_8\dot{v}_1 = 0, \\ &\nu_1 - \lambda_{1,0} = \nu_2 - \lambda_{2,0} = \nu_3 - \lambda_{3,0} = \nu_4 - \lambda_{4,0} = 0, \\ &\lambda_{1,1} = \nu_6 + \lambda_{2,1} = \nu_7 + \lambda_{3,1} = \nu_8 + \lambda_{4,1} = 0. \end{aligned} \tag{23}$$

Eliminating the ν_k, the following conditions remain:

$$\lambda_1 = 0 \qquad \lambda_2\dot{y} + \lambda_3\dot{u} + \lambda_4\dot{v} = 1, \tag{24}$$

both to be satisfied at $t = t_1$. The first of these conditions shows that $a = 0$ and hence that

$$\tan\theta = \frac{c}{b}t + \frac{d}{b}, \tag{25}$$

that is, $\tan\theta$ is linear in t.

Given f explicitly as a function of t, θ is determinable from Eq. (25) and it follows that the Eqs. (14) can be integrated to yield equations of the trajectory. The four constants of integration must be chosen so that the four boundary conditions at $t = 0$, viz. $x = y = u = v = 0$, are satisfied. There are three boundary conditions to be satisfied at $t = t_1$. These suffice to fix the unknown ratios c/b, d/b and the duration of the maneuver t_1. The remaining condition (24) has no practical significance; it serves to determine b, c, d, individually.

In the particular case when f is constant, if $\alpha = -c/b$ and θ_0, θ_1 are the initial and final values of the thrust angle respectively, it can be shown that the three conditions just referred to may be written in the form

$$\begin{aligned} \alpha U &= f\log\left\{\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta_1 + \sec\theta_1}\right\}, \\ \frac{f}{g} &= \frac{\cos\frac{1}{2}(\theta_0 - \theta_1)}{\sin\frac{1}{2}(\theta_0 + \theta_1)}, \\ 2\alpha^2 Y + \alpha U &= f(\sin\theta_0 - \sin\theta_1)\sec\theta_0\sec\theta_1. \end{aligned} \tag{26}$$

Equations (26) can be solved numerically for α, θ_0, θ_1 when g, U, Y are given and the trajectory is then completely determined. If $U = 5$ miles/sec, $Y = 51$ miles, it will be found that

$$\alpha = 2.5 \times 10^{-3} \qquad \theta_0 = 30° \qquad \theta_1 = 0°.$$

Employing Eq. (25), it is immediately deducible from these results that $t_1 = 231$ sec.

IV. Optimal Rocket Trajectories in Vacuo

The general astronautical problem of optimizing a rocket trajectory with respect to the propellant expenditure, may be expressed mathematically thus: D and A are two given points in a given gravitational field; the velocity of a rocket being specified at these points, it is required to connect them by a curve along which the rocket can be navigated with minimum expenditure of propellant. Thus D might be a given launching point on the earth's surface or a point on a circular satellite orbit about this body; in the first case the specified rocket velocity at D would be the velocity of the earth in its orbit compounded with a velocity due to its rotation; in the second case an additional velocity due to the motion in the satellite orbit would also have to be taken into account. A might be a point on the Martian surface or on a satellite orbit about this planet. The times of departure and/or arrival may, or may not, be previously fixed. If any part of the trajectory is immersed in a planetary atmosphere, aerodynamic forces will operate upon the rocket and the aerodynamic characteristics of the vehicle must be supposed known. In this Section we shall assume that all such forces are negligible and that the only forces acting upon the rocket are gravity and the motor thrust.

If $0x_1x_2x_3$ is an inertial frame of reference consisting of three orthogonal axes and if f_i $(i = 1, 2, 3)$ denote the three components along these axes of the gravitational attraction per unit mass at the point (x_1, x_2, x_3) at the time t, we shall assume that the f_i are known functions of t, t_0, t_1, κ_k, x_i, where t_0 is the instant of departure, t_1 is the instant of arrival and the κ_k $(k = 1, 2, \ldots, q)$ are parameters whose physical significance will change from problem to problem. Thus

$$f_i = f_i(t, t_0, t_1, \kappa_k, x_1, x_2, x_3). \tag{27}$$

The equations of motion of a rocket moving in the field can be written

$$\phi_i = \dot{v}_i - \frac{c}{M}\beta l_i - f_i = 0, \tag{28}$$

$$\phi_{i+3} = \dot{x}_i - v_i = 0, \tag{29}$$

$$\phi_7 = \dot{M} + \beta = 0, \tag{30}$$

where the v_i are velocity components, c is the jet velocity, the l_i are the direction cosines of the thrust direction, β is the rate of propellant expenditure and $i = 1, 2, 3$. The identity

$$\phi_8 = l_1^2 + l_2^2 + l_3^2 - 1 = 0 \tag{31}$$

must also be added to the constraints (28)–(30).

For any particular rocket, β will be positive and bounded and thus $0 \leqq \beta \leqq B$. Following Miele [4], we shall make β a monotonic increasing function of a parameter α (of no particular physical significance), such that as α traverses the range $(-\infty, \infty)$, β increases from 0 to B. The functional relationship will be chosen so that $d\beta/d\alpha = 0$ for sufficiently large positive or negative α. Thus, the vanishing of $d\beta/d\alpha$ will imply that either $\beta = 0$ or $\beta = B$, i.e. that the motors are either shut down or are operating at maximum thrust. During the maneuver, α will be some function of t.

The positions of D and A and the velocity of the rocket at these points will be supposed specified, leading to boundary conditions in the form

$$\psi_i = x_{i0} - d_i = 0, \tag{32}$$

$$\psi_{i+3} = x_{i1} - a_i = 0, \tag{33}$$

$$\psi_{i+6} = v_{i0} - D_i = 0, \tag{34}$$

$$\psi_{i+9} = v_{i1} - A_i = 0, \tag{35}$$

where $i = 1, 2, 3$, and d_i and a_i are the coordinates of the points of departure and arrival respectively and the D_i and A_i are the components of the rocket velocity at these points respectively. The instants of departure and arrival may, or may not, be specified. If they are, then additional boundary conditions exist, viz.

$$\psi_{13} = t_0 - T_0 = 0, \tag{36}$$

$$\psi_{14} = t_1 - T_1 = 0. \tag{37}$$

All the constraints have now been stated. There are eleven functions of t subject to these constraints but otherwise arbitrary, viz., the x_i, the v_i, the l_i, M, α. These will be chosen so as to minimize the characteristic velocity of the maneuver, viz.

$$V = c \log(M_0/M_1). \tag{38}$$

The Lagrange function F takes the form

$$F = \lambda_i\left(\dot{v}_i - \frac{c}{M}\beta l_i - f_i\right) + \lambda_{i+3}(\dot{x}_i - v_i) + \lambda_7(\dot{M} + \beta) + \lambda_8(l_1^2 + l_2^2 + l_3^2 - 1), \tag{39}$$

the summation convention being operative. The eleven Euler equations are accordingly

$$\dot{\lambda}_i = -\lambda_{i+3}, \tag{40}$$

$$\dot{\lambda}_{i+3} = -\lambda_j \frac{\partial f_j}{\partial x_i}, \tag{41}$$

$$0 = -\frac{c}{M}\beta\lambda_i + 2\lambda_8 l_i, \tag{42}$$

$$\dot{\lambda}_7 = \frac{c}{M^2}\beta\lambda_i l_i, \tag{43}$$

$$0 = \left(-\frac{c}{M}\lambda_i l_i + \lambda_7\right)\frac{d\beta}{d\alpha}, \tag{44}$$

where i and j range over (1, 2, 3).

From Eqs. (40), (41) it follows that the λ_i satisfy the three equations

$$\ddot{\lambda}_i = \lambda_j \frac{\partial f_j}{\partial x_i}. \tag{45}$$

Relative to rectangular Cartesian frames, the f_j are known to be the components of a vector. It follows that, relative to such frames, $\partial f_j/\partial x_i$ are the components of a second rank tensor and Eq. (45) is accordingly a vector equation. The quantities λ_i can therefore be treated as the components of a vector, which will be referred to as *the primer* [10].

From the Eqs. (42) it follows that

$$\lambda_i = \frac{2M}{c\beta}\lambda_8 l_i \qquad (i = 1, 2, 3), \tag{46}$$

i.e. that the primer is always parallel to the thrust direction (unless $\beta = 0$, when there is no thrust).

From Eqs. (44), it may be concluded that

$$\text{either} \qquad \frac{d\beta}{d\alpha} = 0 \qquad \text{or} \qquad \lambda_7 = \frac{c}{M}\lambda_i l_i. \tag{47}$$

In the second event, by consideration of Eq. (43), it follows that

$$\dot{\lambda}_7 = \frac{\beta}{M}\lambda_7,$$

or

$$\frac{1}{\lambda_7}\dot{\lambda}_7 = -\frac{1}{M}\dot{M}. \tag{48}$$

Integration now yields the result

$$\lambda_7 = \frac{\text{constant}}{M}, \tag{49}$$

and hence, from the second alternative (47),

$$\lambda_i l_i = \text{constant}. \tag{50}$$

But the λ_i are the direction ratios of the thrust, so that

$$l_i = \lambda_i/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}. \tag{51}$$

It now follows from Eq. (50) that

$$\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = \text{constant}, \tag{52}$$

i.e. that the primer is of constant magnitude.

Now, as shown in reference [11], the Eqs. (45) possess no integral in the form of Eq. (52). The first alternative (47) is accordingly forced, imply-

ing that $\beta = 0$ or B, i.e. that the motors must either be shut down or operating at maximum power.

If $\beta = 0$, the rocket coasts under gravity alone, the components of the primer are determined by the Eqs. (45), $\lambda_7 =$ constant, $\lambda_8 = 0$ and the l_i are indeterminate.

If $\beta = B$, the thrust and primer are aligned and hence Eq. (51) is valid. From Eq. (46), it may accordingly be deduced that

$$\lambda_8 = \frac{cB}{2M}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \tag{53}$$

and from Eq. (43) that

$$\dot{\lambda}_7 = \frac{cB}{M^2}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}. \tag{54}$$

Since the optimal trajectory comprises arcs of two types, (a) null-thrust arcs and (b) maximum thrust arcs, it is necessary to consider the Weierstrass–Erdmann corner conditions to be satisfied at the instants the motor is energised or closed down. It will be found that the conditions for the continuity of the $\partial F/\partial \dot{x}_i$ are that the λ_i ($i = 1, 2, 3, 4, 5, 6, 7$) should be continuous at such an instant. Since $\lambda_4 = -\lambda_1$, etc, this implies that the primer and its first derivative are continuous. There is no condition on λ_8. $F - \dot{x}_i\,\partial F/\partial \dot{x}_i$ is continuous if

$$-\lambda_i\left(\frac{c}{M}\beta l_i + f_i\right) - \lambda_{i+3}v_i + \lambda_7\beta$$

is continuous. Since the f_i and v_i are necessarily continuous and the λ_i have been proved continuous, this last condition requires only the continuity of

$$\frac{c}{M}\beta\lambda_i l_i - \lambda_7\beta. \tag{55}$$

Consider an instant at which the motors are brought into operation at maximum thrust. Immediately prior to the instant, $\beta = 0$. Hence, immediately afterwards it is necessary that

$$\frac{c}{M}B\lambda_i l_i - \lambda_7 B = 0,$$

i.e. that

$$\lambda_7 = \frac{c}{M}\lambda_i l_i = \frac{c}{M}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}. \tag{56}$$

However, if the motors are brought into operation at the instant of departure, corner conditions are not appropriate and condition (56) is not applicable.

Similarly, Eq. (56) is to be true at any instant when the motors are closed down, provided this is not the instant of arrival.

Over any portion of the optimal trajectory which lies in a time-invariant gravitational field, the first integral (13b) of the Euler equations is available. This takes the form

$$-\lambda_i\left(\frac{c}{M}\beta l_i + f_i\right) - \lambda_{i+3}v_i + \lambda_7\beta + \lambda_8(l_1^2 + l_2^2 + l_3^2 - 1) = \text{constant}. \tag{57}$$

Making use of the constraints (28)–(31) and the Euler equations (40), (41), this can be written

$$\lambda_i f_i - \dot{\lambda}_i v_i + \beta\left(\frac{c}{M}\lambda_i l_i - \lambda_7\right) = \text{constant}. \tag{58}$$

Condition (56) being satisfied upon entry into or exit from a phase of maximum thrust, it follows that the quantity

$$\beta\left(\frac{c}{M}\lambda_i l_i - \lambda_7\right)$$

is continuous through the value zero at such instants. λ_i, $\dot{\lambda}_i$, v_i and f_i are also continuous. The constant appearing in the right-hand member of Eq. (58) accordingly takes the same value over the whole of the optimal trajectory.

Consider now the constraints (32)–(37) imposed upon the conditions at the end points. The function H is first formed thus:

$$H = c\log\frac{M_0}{M_1} + \nu_i(x_{i0} - d_i) + \nu_{i+3}(x_{i1} - a_i) + \nu_{i+6}(v_{i0} - D_i) + \nu_{i+9}(v_{i1} - A_i) + \nu_{13}(t_0 - T_0) + \nu_{14}(t_1 - T_1), \tag{59}$$

where $\nu_{13} = 0$ if the time of departure is not specified and $\nu_{14} = 0$ if the time of arrival is not specified. Conditions (8)–(12) can now be written down, as shown:

$$\nu_{13} + \nu_i\dot{x}_{i0} + \nu_{i+6}\dot{v}_{i0} + \frac{c}{M_0}\dot{M}_0 - \int_{t_0}^{t_1}\lambda_i\frac{\partial f_i}{\partial t_0}\,dt = 0, \tag{60}$$

$$\nu_{14} + \nu_{i+3}\dot{x}_{i1} + \nu_{i+9}\dot{v}_{i1} - \frac{c}{M_1}\dot{M}_1 - \int_{t_0}^{t_1}\lambda_i\frac{\partial f_i}{\partial t_1}\,dt = 0, \tag{61}$$

$$\nu_i - \lambda_{i+3,0} = 0 \qquad \nu_{i+6} - \lambda_{i,0} = 0, \tag{62}$$

$$\nu_{i+3} + \lambda_{i+3,1} = 0 \qquad \nu_{i+9} + \lambda_{i,1} = 0, \tag{63}$$

$$\frac{c}{M_0} - \lambda_{7,0} = 0 \qquad -\frac{c}{M_1} + \lambda_{7,1} = 0, \tag{64}$$

$$\int_{t_0}^{t_1} \lambda_i \frac{\partial f_i}{\partial \kappa_k} dt = 0. \tag{65}$$

If the times of departure and arrival are specified, conditions (60)–(64) serve only to determine the ν_i and do not provide additional information concerning the optimal trajectory. If, however, the time of departure is to be chosen so that the propellant expenditure is minimized, then $\nu_{13} = 0$ and, by eliminating ν_i, ν_{i+6} between Eqs. (60) and (62), a nontrivial end condition upon the optimal trajectory is found. This is

$$\lambda_i \dot{v}_i + \lambda_{i+3} \dot{x}_i + \frac{c}{M} \dot{M} = \int_{t_0}^{t_1} \lambda_i \frac{\partial f_i}{\partial t_0} dt \tag{66}$$

at the point of departure. Similarly, if the instant of arrival is not given, the following condition is to be satisfied upon arrival:

$$\lambda_i \dot{v}_i + \lambda_{i+3} \dot{x}_i + \frac{c}{M} \dot{M} = -\int_{t_0}^{t_1} \lambda_i \frac{\partial f_i}{\partial t_1} dt. \tag{67}$$

In view of Eqs. (30), (40), these conditions can be written

$$\lambda_i \dot{v}_i - \dot{\lambda}_i v_i - \frac{c}{M} \beta = \int_{t_0}^{t_1} \lambda_i \frac{\partial f_i}{\partial t_0} dt \tag{68}$$

upon departure and

$$\lambda_i \dot{v}_i - \dot{\lambda}_i v_i - \frac{c}{M} \beta = -\int_{t_0}^{t_1} \lambda_i \frac{\partial f_i}{\partial t_1} dt \tag{69}$$

upon arrival.

λ_7 is constant along an arc of null-thrust and is determined by Eq. (54) over an arc of maximum thrust. Equations (64) show that at the end points λ_7 must satisfy the condition

$$\lambda_7 = c/M. \tag{70}$$

The q Eqs. (65) serve to determine the optimal values of the κ_k.

This completes the set of conditions necessarily satisfied by an optimal trajectory. The simpler form to which these conditions reduce in a case of prime importance for the subsequent theory will be investigated in the next Section.

V. The Case of Impulsive Thrusts

In very many of the problems which will be investigated later, although B is finite, the thrust duration necessary to effect the required velocity changes will be small by comparison with the times spent coasting along the null-thrust arcs (a few minutes by comparison with hundreds of days).

It will accordingly be possible, with very great accuracy, to regard the thrust as impulsive and to neglect the rocket motion during a phase of maximum thrust, i.e. the optimal trajectory will be taken to comprise null-thrust arcs meeting at junction points at which the impulses are applied.

On account of the relatively short duration of a thrust phase, the components of the primer and their first derivatives as determined by Eqs. (45) will remain very nearly constant over this period and we shall neglect any variation in these quantities over such a phase. But it has been proved that these quantities are all continuous upon entry into and exit from such a phase. It follows that they are continuous across a junction point from the end of one null-thrust arc to the commencement of the next. This is the first set of simplified conditions.

Suppose that a phase of maximum thrust commences at $t = h$ and terminates at $t = k$. Then, integrating Eq. (54) over the interval (h, k), it will be found that during this phase λ_7 increases by

$$\begin{aligned} \Delta\lambda_7 &= -c \int_h^k \frac{\dot{M}}{M^2} \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}\, dt, \\ &= \Delta\left\{\frac{c}{M} \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}\right\} - c \int_h^k \frac{\lambda_i \dot{\lambda}_i}{M\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} dt, \end{aligned} \tag{71}$$

where an integration by parts has been carried out to yield the Eq. (71). But, since condition (56) is to be satisfied at each end of the interval (h, k) (provided neither is an end point of the trajectory), it follows from Eq. (71) that

$$\int_h^k \frac{\lambda_i \dot{\lambda}_i}{M\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} dt = 0. \tag{72}$$

Neglecting the variation in the λ_i, $\dot{\lambda}_i$ during this phase, condition (72) becomes

$$\frac{\lambda_i \dot{\lambda}_i}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \int_h^k \frac{dt}{M} = 0. \tag{73}$$

The integral is clearly nonzero and thus it follows that

$$\lambda_i \dot{\lambda}_i = 0, \tag{74}$$

at a junction point.

The above argument is invalidated if the point at which the impulse is applied is a terminal of the trajectory and condition (74) is not necessarily satisfied at such a point.

If an impulsive thrust is applied at the point of departure, during this short phase the variation of λ_7 is governed by Eq. (54). Neglecting the

variation in the components of the primer during this phase, Eq. (54) can be integrated over the duration of the thrust to yield

$$\Delta\lambda_7 = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}\,\Delta(c/M). \tag{75}$$

But, at the commencement of the phase, λ_7 is given by Eq. (70) and at its termination by Eq. (56). It follows that at the commencement of the phase

$$\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = 1. \tag{76}$$

Since the components of the primer are direction ratios for the thrust when the motors are operating, this implies that they are, in fact, direction cosines.

If the motors are not operated at the point of departure, λ_7 remains constant at the value given by Eq. (70) until the first junction point is reached when condition (56) is applicable. This implies that Eq. (76) is valid at this junction and the primer components are direction cosines for the impulsive thrust there. At the termination of the first impulsive thrust, Eq. (76) is still valid since the primer components will not have altered appreciably since its inception. But condition (56) is to be satisfied when the motors are closed down. Hence $\lambda_7 = c/M$ at this instant and the above argument can be repeated to show that Eq. (76) is true at the second junction point. Further repetition of this line of argument proves that the result (76) is true at all junction points and condition (70) is then automatically satisfied at the point of arrival.

It follows from the first set of constraints (28) that

$$\lambda_i\dot{v}_i = \frac{c}{M}\beta\lambda_i l_i + \lambda_i f_i = \frac{c}{M}\beta + \lambda_i f_i, \tag{77}$$

since, on a null-thrust arc, $\beta = 0$ and, at a junction point, $\lambda_i = l_i$. Condition (68) can therefore be written

$$\lambda_i f_i - \dot{\lambda}_i v_i = \int_{t_0}^{t_1} \lambda_i \frac{\partial f_i}{\partial t_0}\,dt \tag{78}$$

and is to be satisfied at the point of departure if t_0 is not specified. Similarly, it may be shown from Eq. (69) that

$$\lambda_i f_i - \dot{\lambda}_i v_i = -\int_{t_0}^{t_1} \lambda_i \frac{\partial f_i}{\partial t_1}\,dt \tag{79}$$

at the point of arrival if t_1 is not given.

Over null-thrust arcs passing through a time-invariant gravitational field, the first integral of the Euler equations (58) is available and takes the form

$$\lambda_i f_i - \dot{\lambda}_i v_i = \text{constant}, \tag{80}$$

the constant being the same on any two arcs not separated by a phase

during which the rocket is acted upon by a time-variable field. If the field components f_i are independent of t_0 and t_1 as well as being time-invariant, conditions (78), (79) show that the constant right-hand member of Eq. (80) is zero.

To summarize, if the thrust phases are effectively impulsive, the components of the primer must satisfy Eqs. (45) and, at each junction point,

$$\begin{aligned} &\text{(i) } \lambda_i,\ \dot{\lambda}_i \text{ must be continuous,}\\ &\text{(ii) } \lambda_i\dot{\lambda}_i = 0,\\ &\text{(iii) } \lambda_i \text{ must be direction cosines for the thrust.}\end{aligned} \tag{81}$$

These conditions imply that, at a junction point, the primer and its derivative are continuous and that these vectors are orthogonal.

If an impulsive thrust is applied at a terminal, condition (ii) is not effective at that point.

In addition, if the times of departure and/or arrival are not predetermined, conditions (78) and (79) must be satisfied at the appropriate terminal.

If the gravitational field is time-invariant, Eq. (80) is valid over all null-thrust arcs. Condition (ii) then follows from the circumstance that the right-hand member of this equation assumes the same constant value over two adjacent arcs. For, λ_i, $\dot{\lambda}_i$, f_i being continuous at a junction point, it follows that, if v_i^-, v_i^+ are the velocity components immediately prior and immediately subsequent to the impulsive thrust applied at such a point, then

$$\dot{\lambda}_i v_i^- = \dot{\lambda}_i v_i^+.$$

But

$$v_i^+ - v_i^- \propto l_i = \lambda_i.$$

Hence condition (ii) follows.

The conditions of this Section will be found derived by an alternative method in reference [12].

VI. The Primer on a Keplerian Arc

Over the greater part of an interplanetary trajectory, the attraction of a single body will be paramount. Under these conditions, an arc of null-thrust is a conic with the center of attraction at its focus. Before the theory of the previous Section can be applied to the problem of determining an optimal interplanetary trajectory, it is accordingly necessary to calculate the form taken by the primer along such an arc.

Let rectangular axes $0xyz$ form a nonrotating reference frame and be chosen such that 0 is the center of attraction and $0xy$ is the plane of the conic orbit. Let (r, θ) be polar coordinates of a point P on the conic orbit

measured as indicated in Fig. 1. The axes will be assumed chosen so that θ increases as P advances along the orbit. $0XYz$ constitutes a secondary frame of reference, which rotates about $0z$ with angular velocity $\dot{\theta}$ such

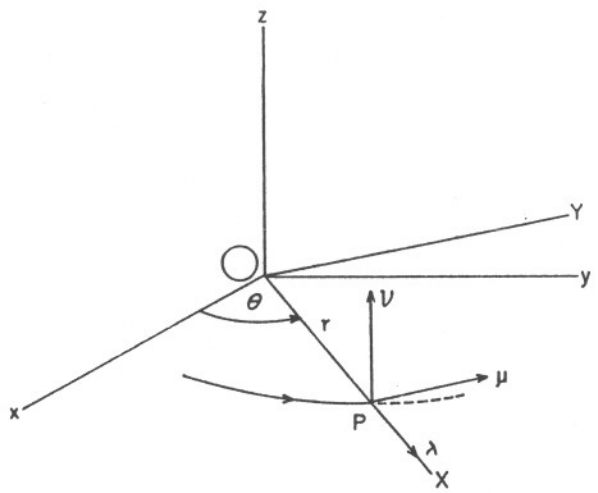

FIG. 1. Frames of reference.

that P remains on $0X$ during its motion. At any point (r, Y, z) in the secondary frame, the gravitational attraction has components

$$f = -\frac{\gamma r}{(r^2 + Y^2 + z^2)^{3/2}} \qquad g = -\frac{\gamma Y}{(r^2 + Y^2 + z^2)^{3/2}} \qquad h = -\frac{\gamma Z}{(r^2 + Y^2 + z^2)^{3/2}}, \tag{82}$$

along the axes of the frame, γ being a constant for the field. Differentiating these components partially with respect to r, Y, z and then putting $Y = z = 0$, it will be found that at P

$$\begin{aligned} &\frac{\partial f}{\partial r} = \frac{2\gamma}{r^3} \qquad \frac{\partial f}{\partial Y} = 0 \qquad \frac{\partial f}{\partial z} = 0 \\ &\frac{\partial g}{\partial r} = 0 \qquad \frac{\partial g}{\partial Y} = -\frac{\gamma}{r^3} \qquad \frac{\partial g}{\partial z} = 0 \\ &\frac{\partial h}{\partial r} = 0 \qquad \frac{\partial h}{\partial Y} = 0 \qquad \frac{\partial h}{\partial z} = -\frac{\gamma}{r^3}. \end{aligned} \tag{83}$$

It follows that, if (λ, μ, ν) are the components of the primer at P relative to the secondary frame (see Fig. 1), the corresponding components of the right-hand member of Eq. (45) relative to this frame are

$$\frac{2\gamma}{r^3}\lambda \qquad -\frac{\gamma}{r^3}\mu \qquad -\frac{\gamma}{r^3}\nu, \tag{84}$$

respectively.

The left-hand member of Eq. (45) will be written $D^2\mathbf{p}$, where $\mathbf{p}$ is the primer vector and D denotes the rate of change operator relative to a non-rotating frame (e.g. $0xyz$). In this Section, dots will be employed to denote rates of change relative to the rotating frame $0XYz$. Then, if $\mathbf{j}$ is the unit vector in the direction $0z$,

$$D^2\mathbf{p} = \ddot{\mathbf{p}} + 2\dot{\theta}\mathbf{j} \times \dot{\mathbf{p}} + \ddot{\theta}\mathbf{j} \times \mathbf{p} + \dot{\theta}^2(\mathbf{j}\cdot\mathbf{p}\mathbf{j} - \mathbf{p}). \tag{85}$$

Introducing a vector $\mathbf{q}$ such that

$$\mathbf{p} = r\mathbf{q}, \tag{86}$$

the identity (85) takes the form

$$D^2\mathbf{p} = r\ddot{\mathbf{q}} + 2\dot{r}\dot{\mathbf{q}} + (\ddot{r} - r\dot{\theta}^2)\mathbf{q} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{j} \times \mathbf{q} + 2r\dot{\theta}\mathbf{j} \times \dot{\mathbf{q}} + r\dot{\theta}^2\mathbf{j}\cdot\mathbf{q}\mathbf{j}. \tag{87}$$

Employing the polar equations of motion of the point P, viz.

$$\ddot{r} - r\dot{\theta}^2 = -\gamma/r^2 \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0, \tag{88}$$

Eq. (87) reduces to the form

$$D^2\mathbf{p} = r\ddot{\mathbf{q}} + 2\dot{r}\dot{\mathbf{q}} + 2r\dot{\theta}\mathbf{j} \times \mathbf{q} + r\dot{\theta}^2\mathbf{j}\cdot\mathbf{q}\mathbf{j} - \frac{\gamma}{r^2}\mathbf{q}. \tag{89}$$

If primes denote differentiations with respect to θ and relative to the rotating frame, then

$$\dot{\mathbf{q}} = \dot{\theta}\mathbf{q}', \tag{90}$$

$$\ddot{\mathbf{q}} = \dot{\theta}^2\mathbf{q}'' + \ddot{\theta}\mathbf{q}', \tag{91}$$

and hence the identity (89) is equivalent to

$$\begin{aligned} D^2\mathbf{p} &= r\dot{\theta}^2\mathbf{q}'' + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{q}' + 2r\dot{\theta}\,\mathbf{j} \times \mathbf{q}' + r\dot{\theta}^2\mathbf{j}\cdot\mathbf{q}\mathbf{j} - \frac{\gamma}{r^2}\mathbf{q}, \\ &= r\dot{\theta}^2(\mathbf{q}'' + 2\mathbf{j} \times \mathbf{q}' + \mathbf{j}\cdot\mathbf{q}\mathbf{j}) - \frac{\gamma}{r^2}\mathbf{q}. \end{aligned} \tag{92}$$

It is known that, for motion along a Keplerian arc

$$r^2\dot{\theta} = \sqrt{\gamma l}, \tag{93}$$

where l is the semilatus rectum of the orbit. It follows that the identity (92) can finally be written in the form

$$D^2\mathbf{p} = \frac{\gamma l}{r^3}\left(\mathbf{q}'' + 2\mathbf{j} \times \mathbf{q}' + \mathbf{j}\cdot\mathbf{q}\mathbf{j} - \frac{r}{l}\mathbf{q}\right). \tag{94}$$

If $\mathbf{q}$ has components (u, v, w) in the directions $0X$, $0Y$, $0z$ respectively, then by Eqs. (86),

$$\lambda = ru \qquad \mu = rv \qquad \nu = rw, \tag{95}$$

and the components of the right-hand member of Eq. (45) along these axes [see Eq. (84)] are accordingly

$$\frac{2\gamma}{r^2}u \qquad -\frac{\gamma}{r^2}v \qquad -\frac{\gamma}{r^2}w. \tag{96}$$

Equating these to the corresponding components of $D^2\mathbf{p}$ as found from the identity (94), the following equations for (u, v, w) are obtained:

$$u'' - 2v' - \frac{3r}{l}u = 0, \tag{97}$$

$$v'' + 2u' \qquad = 0, \tag{98}$$

$$w'' + w \qquad = 0. \tag{99}$$

Since the gravitational field is not time dependent, a first integral (80) of this latter set of equations is available. Now

$$\begin{aligned} D\mathbf{p} &= \dot{\mathbf{p}} + \dot{\theta}\mathbf{j} \times \mathbf{p}, \\ &= r\dot{\mathbf{q}} + \dot{r}\mathbf{q} + r\dot{\theta}\mathbf{j} \times \mathbf{q}, \\ &= r\dot{\theta}\mathbf{q}' + \dot{r}\mathbf{q} + r\dot{\theta}\mathbf{j} \times \mathbf{q}. \end{aligned} \tag{100}$$

Also, relative to the axes $0XYz$, $\mathbf{f}$ has components $(-\gamma/r^2, 0, 0)$ and $\mathbf{v}$ has components $(\dot{r}, r\dot{\theta}, 0)$. Equation (80) accordingly takes the form

$$-\frac{\gamma}{r}u - \dot{r}(r\dot{\theta}u' + \dot{r}u - r\dot{\theta}v) - r\dot{\theta}(r\dot{\theta}v' + \dot{r}v + r\dot{\theta}u) = \text{constant},$$

or

$$r\dot{r}\dot{\theta}u' + r^2\dot{\theta}^2v' + \left(\dot{r}^2 + r^2\dot{\theta}^2 + \frac{\gamma}{r}\right)u = \text{constant}. \tag{101}$$

If e is the eccentricity and ϖ is the longitude of perihelion for the orbit being considered,

$$\frac{l}{r} = 1 + e\cos(\theta - \varpi), \tag{102}$$

$$\dot{r} = e\sqrt{\frac{\gamma}{l}}\sin(\theta - \varpi). \tag{103}$$

Employing these equations and Eq. (93), the first integral is brought to the form

$$u'e\sin f(1 + e\cos f) + v'(1 + e\cos f)^2 + (2 + e^2 + 3e\cos f)u = C, \tag{104}$$

where

$$f = \theta - \varpi \tag{105}$$

is the *real anomaly*.

In the particular case $e = 0$ (i.e. circular orbit), $r = l$ (radius of orbit) and Eqs. (97)–(99) reduce to

$$\begin{aligned} u'' - 2v' - 3u &= 0, \\ v'' + 2u' \qquad &= 0, \\ w'' + w \qquad &= 0. \end{aligned} \tag{106}$$

The first integral (104) is, in this case, the integrated form of the second of this set of equations. The general solution of Eqs. (106) is now easily found and it then follows that

$$\begin{aligned} \lambda &= A\cos f + B\sin f + 2C, \\ \mu &= 2B\cos f - 2A\sin f - 3Cf + D, \\ \nu &= E\cos f + F\sin f, \end{aligned} \tag{107}$$

A, B, C, D, E, F, being constants of integration.

In the case $e \neq 0$, Eq. (97) is discarded in favor of the first integral (104). Integration of Eq. (98) yields the result

$$v' = Ae - 2u. \tag{108}$$

Whence, eliminating v' from Eq. (104), it is found that

$$u'\sin f(1 + e\cos f) + u(e - \cos f - 2e\cos^2 f) = C - A(1 + e\cos f)^2. \tag{109}$$

An integrating factor for this equation is $(1 + e\cos f)^{-2}\sin^{-2} f$ and its general solution is accordingly

$$u = (1 + e\cos f)(A\cos f + Be\sin f + CI), \tag{110}$$

where

$$I = \sin f \int \frac{df}{\sin^2 f(1 + e\cos f)^2}. \tag{111}$$

This integral I can be evaluated in terms of elementary functions and the result will be found in reference [10]. If the instants of departure and arrival are not predetermined, $C = 0$ and I is not required. Substituting for u in Eq. (108) and integrating, v is determined in the form

$$v = (1 + e\cos f)\left\{-A\sin f + B(1 + e\cos f) + \frac{D - A\sin f}{1 + e\cos f} + CJ\right\}, \tag{112}$$

where

$$J = \frac{\cot f}{e(1 + e\cos f)} + \frac{1 + e\cos f}{e\sin f}I. \tag{112a}$$

Equation (99) integrates immediately to yield for w,

$$w = E\cos f + F\sin f. \tag{113}$$

It now follows that

$$\begin{aligned} \lambda &= A\cos f + Be\sin f + CI, \\ \mu &= -A\sin f + B(1 + e\cos f) + \frac{D - A\sin f}{1 + e\cos f} + CJ, \\ \nu &= (1 + e\cos f)^{-1}(E\cos f + F\sin f), \end{aligned} \tag{114}$$

a factor l having been absorbed in each of the six constants of integration A, B, C, D, E, F.

The components of $D\mathbf{p}$ in the same three directions will also be required. Since

$$D\mathbf{p} = \dot{\mathbf{p}} + \dot{\theta}\mathbf{j} \times \mathbf{p}, \tag{115}$$

these are

$$\xi = \dot{\lambda} - \dot{\theta}\mu \qquad \eta = \dot{\mu} + \dot{\theta}\lambda \qquad \zeta = \dot{\nu},$$

or

$$\xi = (\lambda' - \mu)\dot{\theta} \qquad \eta = (\mu' + \lambda)\dot{\theta} \qquad \zeta = \nu'\dot{\theta}. \tag{116}$$

In the case of a circular orbit, by substituting for (λ, μ, ν) from Eqs. (107) and putting $\dot{\theta} = \gamma^{1/2}/a^{3/2}$ where a is the orbital radius, these components of $D\mathbf{p}$ are obtained in the form

$$\begin{aligned} \xi &= \frac{\gamma^{1/2}}{a^{3/2}}(A \sin f - B \cos f + 3Cf - D), \\ \eta &= -\frac{\gamma^{1/2}}{a^{3/2}}(A \cos f + B \sin f + C), \\ \zeta &= \frac{\gamma^{1/2}}{a^{3/2}}(-E \sin f + F \cos f). \end{aligned} \tag{117}$$

In the case $e \neq 0$, it will be found that

$$\begin{aligned} \xi &= \frac{\sqrt{\gamma l}}{r^2}\left\{\frac{A \sin f - D}{1 + e \cos f} - B + CK\right\}, \\ \eta &= \frac{\gamma^{1/2}}{l^{3/2}}\left\{-A(e + \cos f) + De \sin f + C \cos f\right\}, \\ \zeta &= \frac{\gamma^{1/2}}{l^{3/2}}\left\{-E \sin f + (e + \cos f)F\right\}. \end{aligned} \tag{118}$$

where

$$K = \frac{e \sin^2 f - \cos f}{e \sin f(1 + e \cos f)^2} - \frac{I}{e} \operatorname{cosec} f.$$

VII. Optimal Transfer Between Coplanar Ellipses

Suppose that a rocket moves in a gravitational field due to a single attracting body and that conditions at the point of departure are such that, until the motors are energized, it will move in an elliptical orbit having semilatus rectum l_1, eccentricity e_1 and longitude of the nearer apse ϖ_1. Suppose that conditions at the point of arrival are to be such that the rocket is there moving in an elliptical orbit (l_2, e_2, ϖ_2). In this Section it will be assumed that these orbits are coplanar and that the time of transit is not predetermined (thus $C = 0$). Then, if it is also assumed that

the transfer trajectory lies in the plane of the two orbits, it is clear that the primer may be taken to lie in this plane and hence that its z-component is zero (i.e. E, F will be set to zero).

Consider a junction point in this plane at which two Keplerian arcs meet. At this point, suppose transfer takes place from the orbit (l', e', ϖ') to the orbit (l, e, ϖ). Let A', B', D' be the values of the integration constants in the expressions (114) appropriate to the first orbit and A, B, D the values appropriate to the second. Then, if ϕ is the angle made by the direction of the thrust T applied at the junction point with the perpendicu-

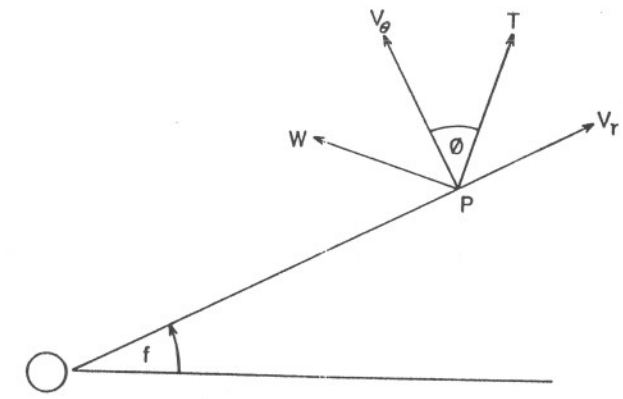

FIG. 2. Impulsive thrust at a junction.

lar to the radius from the center of attraction (Fig. 2), the conditions (81) (i), (iii) require that

$$\begin{aligned} A' \cos f' + B'e' \sin f' &= A \cos f + Be \sin f = \sin \phi, \\ -A' \sin f' + B'(1 + e' \cos f') + \frac{D' - A' \sin f'}{1 + e' \cos f'} \\ = -A \sin f + B(1 + e \cos f) + \frac{D - A \sin f}{1 + e \cos f} &= \cos \phi, \end{aligned} \tag{119}$$

$$l'^{1/2}\left(\frac{A' \sin f' - D'}{1 + e' \cos f'} - B'\right) = l^{1/2}\left(\frac{A \sin f - D}{1 + e \cos f} - B\right),$$

$$l'^{-3/2}\{-A'(e' + \cos f') + D'e' \sin f'\} = l^{-3/2}\{-A(e + \cos f) + De \sin f\},$$

f' and f being the real anomalies on the two orbits at the junction point.

If r is the distance of the junction point from the center of attraction, from Eq. (102) it follows that

$$e' \cos f' = l'/r - 1, \tag{120}$$

$$e \cos f = l/r - 1. \tag{121}$$

Let W be the component in the direction perpendicular to that of the thrust of the rocket's velocity upon arrival at the junction point. Then, if (v_r, v_θ) are the radial and transverse components of this velocity, employing Eqs. (102), (103) and the equation

$$v_\theta = \sqrt{\frac{\gamma}{l}}\,(1 + e\cos f) \tag{122}$$

it follows that

$$\begin{aligned} W &= v_\theta \sin\phi - v_r \cos\phi, \\ &= \frac{\sqrt{\gamma l'}}{r}\sin\phi - e'\sqrt{\frac{\gamma}{l'}}\sin f' \cos\phi. \end{aligned} \tag{123}$$

Hence

$$e' \sin f' = \frac{l'}{r}\tan\phi - W\sqrt{\frac{l'}{\gamma}}\sec\phi. \tag{124}$$

Since the component W is unaffected by the motor thrust,

$$e \sin f = \frac{l}{r}\tan\phi - W\sqrt{\frac{l}{\gamma}}\sec\phi. \tag{125}$$

Eliminating A', B', D', B, D from Eqs. (119) and solving for A, it will be found (employing Eqs. (120)–(125)) that

$$\frac{\gamma^{1/2} e}{l^{1/2}} A = W - \frac{\gamma}{Wr}\sin^2\phi. \tag{126}$$

It follows from the second of Eqs. (119) that

$$Be \sin f = \sin\phi - A\cos f. \tag{127}$$

Substituting for A from Eq. (126) and making use of Eqs. (123), (125), it may be shown that

$$e^2 B = \left(\frac{l}{r} - 1\right)\left(1 + \frac{\gamma^{1/2}}{Wl^{1/2}}\sin\phi\right)\cos\phi + \left(\frac{l}{r}\sin\phi - \frac{Wl^{1/2}}{\gamma^{1/2}}\right)\tan\phi. \tag{128}$$

Writing the fourth of Eqs. (119) in the form

$$B + D = (1 + e\cos f)\cos\phi + (2 + e\cos f)(A\sin f - Be\cos f), \tag{129}$$

and substituting for A, B from Eqs. (126), (127) respectively, it will be found that

$$B + D = -\left\{1 + \frac{\gamma^{1/2}}{Wl^{1/2}}\left(1 + \frac{l}{r}\right)\sin\phi\right\}\cos\phi. \tag{130}$$

Returning now to the two orbits between which transfer is to be effected, if these intersect transfer can take place by means of a single impulsive thrust applied at the point of intersection. The conditions for an optimal transfer will be satisfied by such a maneuver, for the primer

will involve three integration constants A', B', D' on the initial orbit and three A, B, D on the final orbit and it has been shown that the six conditions (119) can be satisfied by appropriate choice of these quantities. The angle ϕ will, of course, be known in this case.

If, however, the orbits do not intersect, two impulses at least must be applied to effect transfer. It is found that, even when the orbits intersect, a two-impulse transfer is often more economical than transfer by a single impulse. However, the theory as developed in Section V, does not provide criteria by reference to which the number of junction points on the absolutely optimal transfer trajectory may be decided. This number can only be found in each particular case by direct comparison of the propellant expenditures relating to the various modes of optimal transfer.

Consider, then, a two-impulse transfer via a transfer orbit (l, e, ϖ) intersecting the given orbits (l_1, e_1, ϖ_1), (l_2, e_2, ϖ_2) in the two junction points. Let r_1, r_2 be the values of r; W_1, W_2 the values of W and ϕ_1, ϕ_2 the values of ϕ at the first and second junction points respectively. If A, B, D are the constants of integration appearing in the expressions for the primer components calculated on the transfer orbit, these quantities must satisfy the Eqs. (126), (128) and (130) at both junction points. Hence, writing

$$r = \frac{1}{s}, \tag{131}$$

$$W = \gamma^{1/2} Z \sin\phi, \tag{132}$$

$$l = \frac{1}{p}, \tag{133}$$

it follows that

$$(Z_1 - s_1/Z_1)\sin\phi_1 = (Z_2 - s_2/Z_2)\sin\phi_2, \tag{134}$$

$$\begin{aligned} &(s_1 - p)(1 + p^{1/2}/Z_1)\cos\phi_1 + (s_1 - Z_1 p^{1/2})\sin\phi_1 \tan\phi_1 \\ &\quad = (s_2 - p)(1 + p^{1/2}/Z_2)\cos\phi_2 + (s_2 - Z_2 p^{1/2})\sin\phi_2 \tan\phi_2, \end{aligned} \tag{135}$$

$$\left(1 + \frac{s_1 + p}{Z_1 p^{1/2}}\right)\cos\phi_1 = \left(1 + \frac{s_2 + p}{Z_2 p^{1/2}}\right)\cos\phi_2. \tag{136}$$

These three conditions, to be satisfied by the optimal transfer orbit will be found derived in references [10] and [11].

Let f_1, f be the real anomalies of the first junction point on the orbit of departure and on the transfer orbit respectively. Then, if θ_1 is the longitude of this junction point

$$f_1 = \theta_1 - \varpi_1 \qquad f = \theta_1 - \varpi. \tag{137}$$

Writing

$$q = e/l, \tag{138}$$

it may now be deduced from Eqs. (120), (121), (124) and (125) that

$$q_1 \cos(\theta_1 - \varpi_1) = s_1 - p_1, \tag{139}$$
$$q_1 \sin(\theta_1 - \varpi_1) = (s_1 - Z_1 p_1^{1/2})\tan \phi_1, \tag{140}$$
$$q \cos(\theta_1 - \varpi) = s_1 - p, \tag{141}$$
$$q \sin(\theta_1 - \varpi) = (s_1 - Z_1 p^{1/2})\tan \phi_1. \tag{142}$$

Similarly, if θ_2 is the longitude of the second junction point,

$$q_2 \cos(\theta_2 - \varpi_2) = s_2 - p_2, \tag{143}$$
$$q_2 \sin(\theta_2 - \varpi_2) = (s_2 - Z_2 p_2^{1/2})\tan \phi_2, \tag{144}$$
$$q \cos(\theta_2 - \varpi) = s_2 - p, \tag{145}$$
$$q \sin(\theta_2 - \varpi) = (s_2 - Z_2 p^{1/2})\tan \phi_2. \tag{146}$$

The eleven Eqs. (134)–(136), (139)–(146) determine the eleven unknowns $s_1, s_2, \theta_1, \theta_2, Z_1, Z_2, \phi_1, \phi_2, p, q, \varpi$ and hence the optimal transfer according to the two-impulse mode.

Suppose that a rocket is moving in an orbit (l, e, ϖ) when an impulsive thrust is applied at a point P transferring it into an orbit (l', e', ϖ'). Referring to Fig. 2, let U be the component of the rocket's velocity in the direction of the thrust immediately prior to the application of the thrust. Then, resolving in the transverse direction, it is clear that

$$v_\theta = U \cos \phi + W \sin \phi \tag{147}$$

and hence that

$$U = \frac{\sqrt{\gamma l}}{r} \sec \phi - W \tan \phi. \tag{148}$$

If U' is this same velocity component immediately after the impulse has been applied, it is proved similarly that

$$U' = \frac{\sqrt{\gamma l'}}{r} \sec \phi - W \tan \phi. \tag{149}$$

It follows that the velocity change caused by the impulse is

$$U' - U = \frac{\gamma^{1/2}}{r} (l'^{1/2} - l^{1/2}) \sec \phi. \tag{150}$$

Making use of the transformations (131), (133), this can be written

$$U' - U = \gamma^{1/2} s (p'^{-1/2} - p^{-1/2}) \sec \phi. \tag{151}$$

The characteristic velocity of the two-impulse maneuver is accordingly

$$\gamma^{1/2} s_1 (p^{-1/2} - p_1^{-1/2}) \sec \phi_1 + \gamma^{1/2} s_2 (p_2^{-1/2} - p^{-1/2}) \sec \phi_2. \tag{152}$$

VIII. Coplanar Terminal Ellipses of Small Eccentricity

Solution of the equations obtained in the last Section governing the optimal orbit of transfer between coplanar ellipses is a process of some

difficulty. If, however, these ellipses have small eccentricity, as shown in reference [13], an approximate solution can be found readily.

Let $(p_1, \epsilon q_1, \varpi_1)$ be parameters of the orbit of departure and $(p_2, \epsilon q_2, \varpi_2)$ the corresponding parameters of the orbit of arrival, ϵ being a small quantity. The eccentricities of these orbits are accordingly $\epsilon q_1/p_1$, $\epsilon q_2/p_2$. The symbols $p, q, \varpi, s_1, \theta_1, s_2, \theta_2, Z_1, Z_2, \phi_1, \phi_2$, having the meanings assigned to them in the previous Section, will here be replaced by $\bar{p}, \bar{q}, \bar{\varpi}$, etc. We shall assume these quantities to be expansible as power series in ϵ, thus:

$$\bar{p} = p + \epsilon p' + \epsilon^2 p'' + \ldots,$$
$$\bar{s}_1 = s_1 + \epsilon s_1' + \epsilon^2 s_1'' + \ldots, \tag{153}$$

etc. (N.B. Primes no longer denote derivatives.)

Substituting these expansions and the parameters of the terminal orbits in Eqs. (134)–(136), (139)–(146) and equating coefficients of like powers of ϵ from the two members of each equation, we obtain from the terms of zero degree in ϵ the following equations:

$$(Z_1 - s_1/Z_1)\sin \phi_1 = (Z_2 - s_2/Z_2)\sin \phi_2, \tag{154}$$

$$(s_1 - p)(1 + p^{1/2}/Z_1)\cos \phi_1 + (s_1 - Z_1 p^{1/2})\sin \phi_1 \tan \phi_1$$
$$= (s_2 - p)(1 + p^{1/2}/Z_2)\cos \phi_2 + (s_2 - Z_2 p^{1/2})\sin \phi_2 \tan \phi_2, \tag{155}$$

$$\left(1 + \frac{s_1 + p}{Z_1 p^{1/2}}\right) \cos \phi_1 = \left(1 + \frac{s_2 + p}{Z_2 p^{1/2}}\right) \cos \phi_2, \tag{156}$$

$$0 = s_1 - p_1, \tag{157}$$
$$0 = (s_1 - Z_1 p_1^{1/2})\tan \phi_1, \tag{158}$$
$$q \cos(\theta_1 - \varpi) = s_1 - p, \tag{159}$$
$$q \sin(\theta_1 - \varpi) = (s_1 - Z_1 p^{1/2})\tan \phi_1, \tag{160}$$
$$0 = s_2 - p_2, \tag{161}$$
$$0 = (s_2 - Z_2 p_2^{1/2})\tan \phi_2, \tag{162}$$
$$q \cos(\theta_2 - \varpi) = s_2 - p, \tag{163}$$
$$q \sin(\theta_2 - \varpi) = (s_2 - Z_2 p^{1/2})\tan \phi_2. \tag{164}$$

If Eqs. (158), (162) are satisfied by taking

$$s_1 = Z_1 p_1^{1/2} \qquad s_2 = Z_2 p_2^{1/2}, \tag{165}$$

it will be found that this leads to unacceptable conditions on p_1 and p_2. The only possibility is that

$$\tan \phi_1 = \tan \phi_2 = 0. \tag{166}$$

It now follows from Eqs. (160), (164) that the angles $\theta_1 - \varpi$, $\theta_2 - \varpi$ are each 0 or π. However, if both these angles are zero or both π, Eqs. (159), (163) are contradictory. Hence, we shall take

$$\theta_1 - \varpi = 0 \qquad \theta_2 - \varpi = \pi. \tag{167}$$

Equations (159), (163) then yield the results

$$p = \tfrac{1}{2}(s_1 + s_2) = \tfrac{1}{2}(p_1 + p_2), \tag{168}$$

$$q = \tfrac{1}{2}(s_1 - s_2) = \tfrac{1}{2}(p_1 - p_2), \tag{169}$$

indicating that it is being assumed that $p_1 > p_2$, i.e. that the orbit of departure lies within the orbit of arrival. If the contrary is true, it is necessary to take

$$\theta_1 - \varpi = \pi \qquad \theta_2 - \tilde{\omega} = 0. \tag{170}$$

Since ϕ_1, ϕ_2 are each 0 or π, Eq. (154) is clearly satisfied. The remaining Eqs. (155), (156) determine Z_1, Z_2. Before solving for these quantities, the zero order solution will be studied in detail.

If $\epsilon = 0$, the orbits of departure and arrival are circles and the zero order solution represents the exact solution to the problem of optimal transfer between them. Equations (167) indicate that the impulsive thrusts are applied at the two apses of the transfer ellipse, which is accordingly tangential to the circular orbits at these points. This mode of transfer between two circular orbits via a *cotangential ellipse* was discovered by

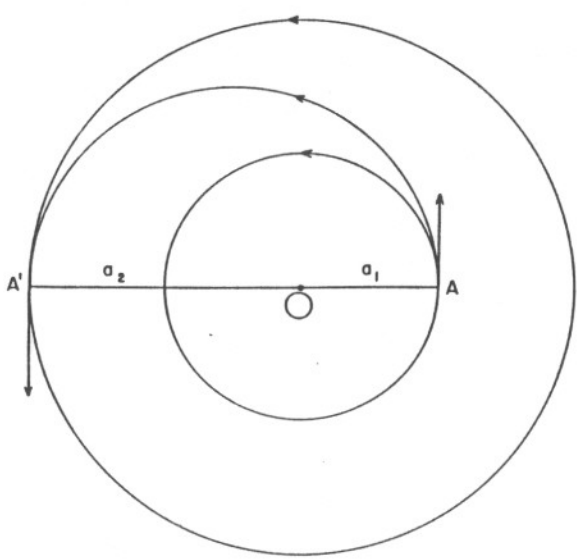

FIG. 3. The Hohmann ellipse.

Hohmann [14] and the ellipse is named after him. It is shown in Fig. 3. If the transfer is to take place from the smaller to the larger circle, the thrusts applied at the apses A, A′ are in the direction of motion at these points and $\phi_1 = \phi_2 = 0$. For transfer from the larger to the smaller circle,

$\phi_1 = \phi_2 = \pi$. The orientation of the major axis AA′ of the transfer ellipse is indeterminate, i.e. ϖ can be given any value and Eqs. (167) [or (170)] then determine θ_1, θ_2. If a_1, a_2 are the radii of the two circular orbits ($a_1 < a_2$), then $p_1 = 1/a_1$, $p_2 = 1/a_2$ and, from Eqs. (168), (169), it follows that the semi-latus rectum and eccentricity of the transfer orbit are given by

$$l = \frac{1}{p} = \frac{2a_1a_2}{a_1 + a_2}, \tag{171}$$

$$e = \frac{q}{p} = \frac{a_2 - a_1}{a_2 + a_1}. \tag{172}$$

Whence, the semimajor axis of this orbit is given by

$$a = \tfrac{1}{2}(a_1 + a_2). \tag{173}$$

This also follows from inspection of Fig. 3.

From Eq. (152) it is now found that the characteristic velocity of this maneuver is

$$\sqrt{\frac{\gamma}{a_1}}\left(\sqrt{\frac{2}{1+\beta}} - 1\right) + \sqrt{\frac{\gamma}{a_2}}\left(1 - \sqrt{\frac{2\beta}{1+\beta}}\right), \tag{174}$$

where

$$\beta = a_1/a_2. \tag{175}$$

Returning to the general case when $\epsilon \neq 0$, putting $\phi_1 = \phi_2 = 0$ (or $\phi_1 = \phi_2 = \pi$ as may be appropriate) in Eqs. (155), (156), these can be reduced to

$$2\sqrt{2} + (p_1 + p_2)^{1/2}\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) = 0, \tag{176}$$

$$\frac{3p_1 + p_2}{Z_1} = \frac{p_1 + 3p_2}{Z_2}. \tag{177}$$

It now follows that

$$Z_1 = -\frac{\sqrt{2}(p_1 + p_2)}{p_1 + 3p_2}, \qquad Z_2 = -\frac{\sqrt{2}(p_1 + p_2)}{3p_1 + p_2}. \tag{178}$$

We proceed now to a consideration of the terms of the first degree in ϵ occurring in the Eqs. (134)–(136), (139)–(146). After some simplification by making use of results just obtained, it will be found that these yield the conditions

$$\left(Z_1 - \frac{p_1}{Z_1}\right)\phi_1' = \left(Z_2 - \frac{p_2}{Z_2}\right)\phi_2', \tag{179}$$

$$(s_1' - p')\left(1 + \frac{p^{1/2}}{Z_1}\right) + (p_1 - p)\left(\frac{p'}{2p^{1/2}Z_1} - \frac{p^{1/2}Z_1'}{Z_1^2}\right)$$
$$= (s_2' - p')\left(1 + \frac{p^{1/2}}{Z_2}\right) + (p_2 - p)\left(\frac{p'}{2p^{1/2}Z_2} - \frac{p^{1/2}Z_2'}{Z_2^2}\right), \quad (180)$$

$$\frac{s_1' + p'}{Z_1 p^{1/2}} - (p_1 + p)\left(\frac{Z_1'}{Z_1^2 p^{1/2}} + \frac{p'}{2Z_1 p^{3/2}}\right)$$
$$= \frac{s_2' + p'}{Z_2 p^{1/2}} - (p_2 + p)\left(\frac{Z_2'}{Z_2^2 p^{1/2}} + \frac{p'}{2Z_2 p^{3/2}}\right), \quad (181)$$

$$\pm q_1 \cos(\varpi - \varpi_1) = s_1', \quad (182)$$
$$q_1 \sin(\varpi - \varpi_1) = (p_1 - Z_1 p_1^{1/2})\phi_1', \quad (183)$$
$$\pm q' = s_1' - p', \quad (184)$$
$$q(\theta_1' - \varpi') = (p_1 - Z_1 p^{1/2})\phi_1', \quad (185)$$
$$\mp q_2 \cos(\varpi - \varpi_2) = s_2', \quad (186)$$
$$-q_2 \sin(\varpi - \varpi_2) = (p_2 - Z_2 p_2^{1/2})\phi_2', \quad (187)$$
$$\mp q' = s_2' - p', \quad (188)$$
$$-q(\theta_2' - \varpi') = (p_2 - Z_2 p^{1/2})\phi_2', \quad (189)$$

taking the upper sign in each case of ambiguity if the transfer is from the inner to the outer ellipse and the lower sign otherwise (this convention is applicable throughout the remainder of this Section).

Division of Eq. (187) by (183) yields

$$-\frac{q_2 \sin(\varpi_2 - \varpi)}{q_1 \sin(\varpi_1 - \varpi)} = \frac{(p_2 - Z_2 p_2^{1/2})\,\phi_2'}{(p_1 - Z_1 p_1^{1/2})\,\phi_1'}. \quad (190)$$

Employing Eq. (179) and substituting for Z_1, Z_2 from Eqs. (178), this may be written

$$-\frac{\sin(\varpi_2 - \varpi)}{\sin(\varpi_1 - \varpi)} = \frac{q_1}{q_2} P\left(\frac{p_2}{p_1}\right), \quad (191)$$

where, in its simplest form,

$$P(x) = \frac{1 + 3x - \sqrt{2}(1 + x)^{3/2}}{3 + x - \sqrt{(2/x)}(1 + x)^{3/2}}. \quad (192)$$

Tan ϖ is determined by the Eq. (191) in the form

$$\tan \varpi = \frac{Q \sin \varpi_1 + \sin \varpi_2}{Q \cos \varpi_1 + \cos \varpi_2}, \quad (193)$$

where

$$Q = \frac{q_1}{q_2} P\left(\frac{p_2}{p_1}\right). \quad (194)$$

It will be found that, very approximately,

$$P(x) = x \quad (195)$$

and hence, to the same degree of approximation,

$$Q = \frac{q_1 p_2}{q_2 p_1} = \frac{e_1}{e_2}. \quad (196)$$

Thus

$$\tan \varpi = \frac{e_1 \sin \varpi_1 + e_2 \sin \varpi_2}{e_1 \cos \varpi_1 + e_2 \cos \varpi_2} \quad (197)$$

approximately. Hence, if $e_1 \gg e_2$, $\varpi = \varpi_1$, whereas if $e_2 \gg e_1$, $\varpi = \varpi_2$. These results may be expressed by saying that *the major axis of the optimal transfer ellipse tends to align itself with the major axis of the terminal ellipse having the greater eccentricity*. This is the *Rule of Attraction of Eccentricities*. In particular, if one terminal orbit is a circle and the other is an ellipse, it will be found that the equations governing the optimal orbit of transfer are easily solved and that the major axes of the transfer and terminal ellipses are exactly aligned and that the transfer ellipse is tangential to both terminals. The details will be found in reference [13].

ϖ being found, Eqs. (182, 186) determine s_1', s_2'. Then, from Eqs. (184), (188), it is found that

$$p' = \pm \tfrac{1}{2}\{q_1 \cos(\varpi - \varpi_1) - q_2 \cos(\varpi - \varpi_2)\}, \quad (198)$$
$$q' = \tfrac{1}{2}\{q_1 \cos(\varpi - \varpi_1) + q_2 \cos(\varpi - \varpi_2)\}. \quad (199)$$

Equations (183, 187) determine ϕ_1', ϕ_2' and Eqs. (185), (189) then yield $(\theta_1' - \varpi')$, $(\theta_2' - \varpi')$. ϖ' is indeterminate at this stage, but can be found, if desired, by consideration of second order terms. Finally, Z_1', Z_2' are determined by Eqs. (180, 181). It is now possible to proceed to a consideration of terms of the second order in ϵ if desired.

From Eq. (152), neglecting terms of the second order in ϵ, it will be found that the characteristic velocity V of the transfer maneuver is given by

$$\gamma^{-1/2} V = \pm \{p_2^{1/2} - p_1^{1/2} + p^{-1/2}(p_1 - p_2)\}$$
$$- \epsilon\{G \cos(\varpi - \varpi_1) + H \cos(\varpi - \varpi_2)\}, \quad (200)$$

where G and H are positive quantities defined by

$$G = \left\{p_1^{-1/2} - \frac{p_1 + 3p_2}{\sqrt{2}(p_1 + p_2)^{3/2}}\right\} q_1,$$
$$H = \left\{p_2^{-1/2} - \frac{3p_1 + p_2}{\sqrt{2}(p_1 + p_2)^{3/2}}\right\} q_2. \quad (201)$$

Equation (193) has two distinct solutions in ϖ differing by an angle π. It is clear from Eq. (200) that the solution which makes the expression

$$G \cos(\varpi - \varpi_1) + H \cos(\varpi - \varpi_2) \tag{202}$$

positive is that which minimizes V.

The first order approximation to the optimal transfer ellipse which has been found may be shown (see reference [13]), by comparison with the results of reference [15], to be identical, to the same order of approximation, with the optimal cotangential ellipse. However, it is not true generally that optimal transfer can be effected via a cotangential ellipse.

Having given a detailed discussion of the problem of optimal transfer employing two impulses, the question arises of whether the introduction of a third impulse might further reduce the propellant expenditure? It has been shown by Hoelker and Silber [16] that, in the case of transfer between two coplanar circles whose radii are in a ratio of greater than 11.94 : 1, there is always a three-impulse mode which is more economical than the two-impulse Hohmann mode. However, a general investigation of this question has not yet been carried out.

IX. Transfer Between Non-coplanar Orbits

This is the general three-dimensional problem and is only now in the process of investigation. If the orbits are circular, it appears that for two-impulse transfer, the junction points lie on the line of intersection of the orbital planes and are at the apses of the transfer ellipse. The plane of this ellipse lies between the planes of the circular orbits.

If the eccentricities of the terminal orbits are small, the optimal two-impulse transfer between them can be found by modification of the optimal maneuver between two circles, the method being similar to that employed in the last Section. The junction points will then be close to the line of intersection of the orbital planes. However, if the angle between these planes is steadily reduced, so that it becomes small, it may be expected that these junction points will move around the two terminal orbits and approach the positions they occupy when the orbits are coplanar. Thus, if both eccentricities and the angle between the orbital planes are small, the two junction points will occupy some intermediate position and the solution cannot be found by applying small corrections to either the "coplanar" solution or the "circular" solution. Unfortunately, this is the case of most practical interest, since the planetary orbits are of this type.

The work reported in reference [17] indicates that, if the angle between the orbital planes is large, a three-impulse transfer will generally prove to be more economical.

X. Escape from and Entry into Circular Orbits

Suppose that a vehicle has been placed in circular orbit about the planet of departure and that it is required to transfer it into the transfer orbit along which it will approach the planet of arrival. Relative to the sun, the velocity of entry into the transfer orbit will be known. The vehicle must be maneuvered in such a way that its ultimate velocity $\mathbf{v}$ of recession from the planet of departure when compounded vectorially with the velocity $\mathbf{u}$ of this planet in its orbit about the sun is equal to the required velocity of entry $\mathbf{w}$ into the transfer orbit. That is,

$$\mathbf{w} = \mathbf{u} + \mathbf{v} \quad \text{or} \quad \mathbf{v} = \mathbf{w} - \mathbf{u}. \tag{203}$$

Neglecting the effect of the sun's field during the escape maneuver and considering only the motion relative to the planet, the problem is to transfer

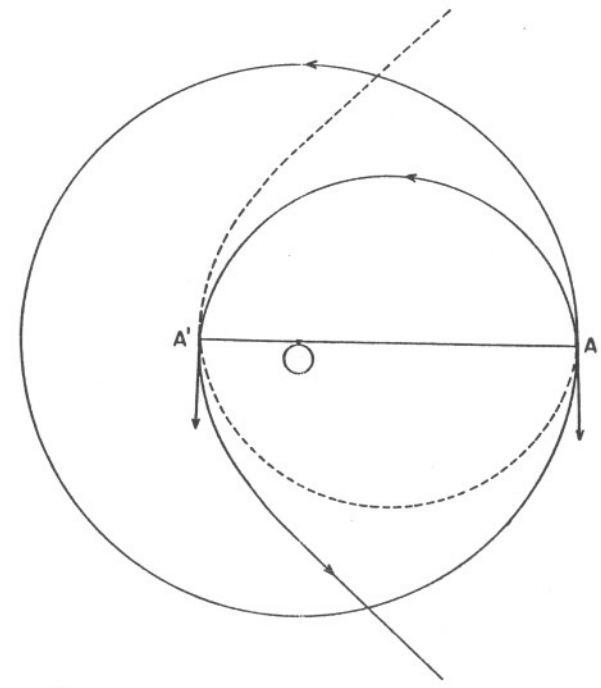

FIG. 4. Transfer from circular to hyperbolic orbit.

the rocket from its circular orbit into a hyperbolic orbit along which it will ultimately recede from the planet with the desired velocity $\mathbf{v}$. The direction of $\mathbf{v}$, together with the center of the planet, determine the plane of the hyperbolic orbit of escape. It will be assumed that the circular orbit has

been chosen to lie in this plane. The theory of Section VIII indicates that optimal transfer between the circular and hyperbolic orbits will be effected via a transfer ellipse whose axis is aligned with that of the hyperbolic orbit and which is tangential at its apses to both the hyperbolic and circular orbits (Fig. 4). Impulsive thrusts are applied parallel to the direction of motion at the apses A, A′, to effect transfer into and out of the elliptical orbit respectively. There are a single infinity of hyperbolas corresponding to the given velocity of recession **v** and it is desirable to choose that one which results in minimum overall expenditure of propellant. A can be chosen to be any point on the circular orbit, so that the direction of **v** can be adjusted as required.

Let OA $= a$ (the radius of the circular orbit) and OA′ $= a'$. Also, let v_3 be the velocity in the hyperbolic orbit at A′. If γ/r^2 is the attraction per unit mass due to the planet, the energy equation for the hyperbolic orbit yields

$$v_3^2 = v^2 + 2\gamma/a, \tag{204}$$

v being the required velocity at infinity on this trajectory. The semimajor axis of the elliptical orbit is $\frac{1}{2}(a + a')$. Hence, if v_1 is the velocity in this orbit at A and v_2 the velocity at A′, the energy equation for motion in this orbit leads to the results

$$v_1^2 = \gamma\left(\frac{2}{a} - \frac{2}{a + a'}\right) = \frac{2\gamma a'}{a(a + a')}, \tag{205}$$

$$v_2^2 = \gamma\left(\frac{2}{a'} - \frac{2}{a + a'}\right) = \frac{2\gamma a}{a'(a + a')}. \tag{206}$$

If v_0 is the velocity in the circular orbit, then

$$v_0^2 = \gamma/a. \tag{207}$$

If, now, $a' < a$ as in Fig. 4, the impulse at A opposes the motion, whereas that at A′ supports it. The characteristic velocity of the maneuver in this case is

$$V = (v_0 - v_1) + (v_3 - v_2). \tag{208}$$

If, however, $a' > a$, both impulses are in the direction of motion and

$$V = (v_1 - v_0) + (v_3 - v_2). \tag{209}$$

Substituting for v_0, v_1, v_2, v_3 from Eqs. (204)–(207), it will be found that

$$\begin{aligned} V &= v_0\{\sqrt{2}(x + \rho^2)^{1/2} - \sqrt{2}(x + 1)^{1/2} + 1\} \qquad x \geqq 1, \\ &= v_0\{\sqrt{2}(x + \rho^2)^{1/2} + \sqrt{2}(1 - x)(1 + x)^{-1/2} - 1\} \qquad x \leqq 1, \end{aligned} \tag{210}$$

where $x = a/a'$ and $\rho = v/\sqrt{2}v_0$. Thus ρ is the ratio of the velocity required at infinity to the escape velocity for a stationary body at any point of the circular orbit.

If V/v_0 be plotted against x for various values of ρ from the formulas (210) (see reference [18]), it appears that this quantity decreases monotonically as $x \to \infty$ if $\rho > 1$. In this case V is minimized by taking a' as small as possible. On the other hand, if $\rho < 1$, V possesses a minimum at $x = 1$. Thus V is then minimized by taking $a' = a$. In this case, therefore, the elliptical transfer orbit is identical with the circular orbit. No impulse is now applied at A and a single impulse effects transfer from the circular into the hyperbolic orbit which is tangential to it.

To summarize the above results: If the velocity desired after escape has been effected is less than the escape velocity at any point of the circular orbit, escape should be effected by application of a single impulse in the direction of the tangent to the circular orbit. If, however, the final velocity of recession is to be greater than this critical value, it is advantageous first to apply a tangential impulse opposing the motion, thus causing the vehicle to approach the center of attraction, and to effect transfer into the hyperbolic orbit at the point of closest approach by means of a second impulse applied in the direction of the motion at this point.

Transferring our attention to the problem of optimal entry into a circular orbit about the planet of arrival, it will be observed that the velocity of approach from infinity will be the vector difference between the velocity of the rocket relative to the sun at the end of the transfer trajectory and the velocity of the planet of arrival in its orbit. The velocity of approach will accordingly be specified both as regards magnitude and direction. However, a small variation in the conditions at launching into the transfer ellipse will result in a large variation of the line of approach to the planet of arrival. This line of approach, therefore, is not to be regarded as determined by the mode of transfer. Throughout the interplanetary journey, course and speed corrections will have to be made to the vehicle as it is observed to deviate from the precalculated track and, as the planet of arrival is approached, such corrections can be utilized to modify the line of approach to that most suited to the purpose. These corrections will not appreciably affect the velocity of approach, which will accordingly be presumed known. Four problems of approach with a given velocity will now be considered. In each case, it is required to select values for the variable quantities which will ensure minimum propellant expenditure during the maneuver.

(i) To enter a circular orbit of given radius, the line of approach being specified.

(ii) To enter a circular orbit of given radius, the line of approach being open to choice.

(iii) The line of approach being given, to enter a circular orbit whose radius is open to choice.

(iv) To enter a circular orbit of arbitrary radius utilizing an arbitrary line of approach.

The line of approach relative to the planet will be an asymptote of a hyperbolic orbit with its focus at the planet's center. The first problem is therefore that of optimal transfer between a given hyperbolic orbit and a given circular one. The optimal maneuver will, of course, be that illustrated in Fig. 4, but carried out in the reverse sense. The characteristic velocity of the maneuver is given by Eqs. (210).

Similarly, the second problem is mathematically identical, apart from the sense of the motion, with that of optimal escape from a given circular orbit when the velocity of ultimate recession is specified, but the line of recession is variable. From what has been proved above, we conclude that, provided the velocity of approach is not greater than the velocity of escape at the level of the circular orbit, the line of approach should be chosen so that the hyperbolic orbit is tangential at its apse to the circular orbit. A single impulsive thrust applied at this apse then effects transfer. If, however, the velocity of approach is greater than that of escape at the circular orbit, the line of approach should be selected so that the apse of the hyperbolic orbit is as close as possible to the center of the planet, e.g. at the outer fringes of its atmosphere. An impulsive thrust is then applied at this apse to effect transfer into an ellipse tangential to both the hyperbolic and circular orbits and a second thrust transfers the rocket from the elliptical into the circular orbit.

In the third problem, employing the notation described earlier in this Section, a' is given and a is to be chosen so as to minimize the characteristic velocity V. Writing $y = a'/a$, the Eqs. (210) take the form

$$\begin{aligned} V &= v_c\{y^{1/2} - \sqrt{2}(1+y)^{1/2} + k\} \qquad y \leqq 1, \\ &= v_c\{\sqrt{2}(y-1)(y+1)^{-1/2} - y^{1/2} + k\} \qquad y \geqq 1, \end{aligned} \tag{211}$$

where $v_c = (\gamma/a')^{1/2}$ is the circular velocity at the apse of the hyperbolic trajectory of approach, and

$$k = (2 + v^2/v_c^2)^{1/2}. \tag{212}$$

v is the velocity of approach from infinity.

It will be found that V is an increasing function of y for all $y \geqq 0$. It follows that y should be made as small as possible and hence that a should be as large as possible. Thus, for a given line of approach, the radius of the circular orbit should be as large as is convenient. As $a \to \infty$, $y \to 0$ and the characteristic velocity approaches a lower bound, viz.

$$\begin{aligned} V = V_0 &= v_c(k - \sqrt{2}), \\ &= (2v_c^2 + v^2)^{1/2} - \sqrt{2}v_c. \end{aligned} \tag{213}$$

Finally, consider the fourth problem. As has just been proved, for each line of approach the propellant expenditure is minimized by transferring into a circular orbit of large radius. The characteristic velocity for such a transfer is given by Eq. (213). Since the line of approach is now variable, so is v_c. But V_0 is a decreasing function of v_c, approaching zero as $v_c \to \infty$. It is accordingly advantageous to choose the line of approach to make v_c as large as possible. This requires that a' shall be as small as possible, i.e. that the approach to the planetary surface shall be as close as possible.

The approach maneuver can therefore be made most economical in propellant by choosing the line of approach so that the planetary surface (or atmosphere, if any) is grazed and the final satellite orbit into which entry is made is of large radius. If R is the radius of the planet (and its atmosphere, if any), $a' \geqq R$ and hence $v_c \leqq (\gamma/R)^{1/2}$. From Eq. (213), it follows that the absolute minimum characteristic velocity of the approach maneuver is

$$V_0 = (2\gamma/R + v^2)^{1/2} - (2\gamma/R)^{1/2}. \tag{214}$$

The impulsive thrusts assumed in the above theory can only be approximated in practice. The problem of optimal escape from a circular orbit, account being taken of the fact that the motor thrust is finite, has also been studied, however, and the details will be found in references [6], and [19]. Only the results will be stated here. Assuming that the motors are energized continuously at maximum thrust until escape has been effected, the propellant expenditure is minimized by displacing the thrust away from the direction of motion by a small angle towards the center of attraction. This angle must be decreased steadily as the maneuver proceeds and is zero at the termination of the maneuver so that the thrust and velocity vectors are then aligned. The maximum value of this "throw-off angle" occurs at the commencement of the maneuver, but is normally not greater than 5°. If the thrust and velocity vectors are maintained in alignment throughout the maneuver, the increase in propellant expenditure is very small, so that, for all practical purposes, this may be regarded as the optimal mode of escape.

XI. Optimal Transfer between Two Planets

A. General Theory

In Section VII the problem of optimal transfer between two elliptical orbits was considered and the results obtained are applicable to the prob-

lem of transfer between two planets if the gravitational fields of these bodies may be neglected. This will be the case if the rocket is rotating initially about the planet of departure in a circular orbit of large radius and it is required that, at the termination of the transfer maneuver, it shall be rotating about the planet of arrival in another circular orbit of large radius, it being assumed that exit from and entry into these circular orbits is achieved in each case by a single impulsive thrust. However, as shown in the previous section, in such circumstances it will be advantageous to make a close approach to each planet and hence their fields should still be allowed for when calculating the interplanetary transfer ellipse of least propellant expenditure. The way in which this can be done will now be explained.

Let f_i^0, f_i^1, f_i^2 $(i = 1, 2, 3)$ be the components of the attractions of the sun, planet of departure and planet of arrival respectively along the rectangular axes of an inertial frame $0x_1x_2x_3$ fixed in the sun. It will be convenient to choose an instant when the planet of departure is at perihelion as the instant when the transfer maneuver commences. This is not to be taken to imply that the motors are necessarily activated at this instant. Let $t = t_0$ be this instant. Then, if t is any later instant, the coordinates of this planet will be expressible as functions of $(t - t_0)$ and hence

$$f_i^1 = f_i^1(x_1, x_2, x_3, t - t_0). \tag{215}$$

Similarly, it will be convenient to suppose that the transfer maneuver terminates at an instant $t = t_1$ when the planet of arrival passes through perihelion and then f_i^2 will be expressible thus:

$$f_i^2 = f_i^2(x_1, x_2, x_3, t - t_1). \tag{216}$$

t_0, t_1 will be supposed freely variable, i.e. it will be assumed that the relative configuration of the planets at the commencement of the maneuver is arbitrary and is to be arranged in such a manner that the transfer is optimal. This may involve a waiting period of some years.

For the sun's field,

$$f_i^0 = f_i^0(x_1, x_2, x_3). \tag{217}$$

The resultant field at any point (x_1, x_2, x_3) and instant t is then given by

$$f_i = f_i^0 + f_i^1 + f_i^2. \tag{218}$$

At $t = t_0$, the rocket will be supposed to be in orbit about the planet of departure or at rest upon its surface. Since the position of the planet is known at this instant, the end conditions upon the rocket will be completely specified. Similarly, it may be assumed that the end conditions at $t = t_1$ are also known.

The general theory of Section V is now applicable to the problem and,

in particular, conditions (78), (79) have to be satisfied at the instants t_0, t_1 respectively. But, it is clear from Eqs. (215), (217), (218) that

$$\partial f_i/\partial t_0 = \partial f_i^1/\partial t_0 = -\partial f_i^1/\partial t, \tag{219}$$

and hence, at $t = t_0$, it is necessary that

$$\lambda_i f_i - \dot{\lambda}_i v_i = -\int_{t_0}^{t_1} \lambda_i \frac{\partial f_i^1}{\partial t}\, dt. \tag{220}$$

Similarly, at $t = t_1$ it is necessary that

$$\lambda_i f_i - \dot{\lambda}_i v_i = \int_{t_0}^{t_1} \lambda_i \frac{\partial f_i^2}{\partial t}\, dt. \tag{221}$$

For the problem we are considering, on a null-thrust arc Eq. (13) takes the form

$$K = \dot{\lambda}_i v_i - \lambda_i f_i. \tag{222}$$

Then, from Eq. (13a),

$$\frac{dK}{dt} = -\lambda_i \frac{\partial f_i}{\partial t}. \tag{223}$$

Since λ_i, $\dot{\lambda}_i$, f_i are continuous across a junction, K is also continuous if

$$\dot{\lambda}_i v_i^- = \dot{\lambda}_i v_i^+,$$

i.e. if

$$\dot{\lambda}_i(v_i^+ - v_i^-) = 0. \tag{224}$$

But the velocity increment is in the direction of the primer at the junction and hence Eq. (224) is equivalent to

$$\lambda_i \dot{\lambda}_i = 0. \tag{225}$$

This is condition (81) (ii) and is satisfied in the case of an optimal maneuver. It follows that K is continuous across a junction and hence, by integrating Eq. (223) over the interval (t_0, t), that

$$K - K_0 = -\int_{t_0}^{t} \lambda_i \frac{\partial f_i}{\partial t}\, dt, \tag{226}$$

where $K_0 = K(t_0)$.

If t is any instant prior to the approach of the rocket to the planet of arrival, f_i^2 will be negligible and Eq. (226) is equivalent to

$$K - K_0 = -\int_{t_0}^{t} \lambda_i \frac{\partial f_i^1}{\partial t}\, dt, \tag{227}$$

f_i^0 not being explicitly dependent upon t. Condition (220) may be written

$$K_0 = \int_{t_0}^{t_1} \lambda_i \frac{\partial f_i^1}{\partial t}\, dt. \tag{228}$$

It now follows that

$$K = \int_{t}^{t_1} \lambda_i \frac{\partial f_i^1}{\partial t}\, dt. \tag{229}$$

But f_i^1 is negligible over the interplanetary transfer ellipse and hence

$$K = 0 \tag{230}$$

over this trajectory.

Condition (221) leads, by a similar argument, to the same result and hence does not represent an independent condition.

Since, over the interplanetary transfer ellipse, the gravitational field is time invariant, it is known that a first integral of the Euler equations exists on this arc in the form of Eq. (80). It has now been proved that the constant right-hand member of this equation is zero. It will be recalled that this was also the case when the planetary fields were altogether neglected. However, K can no longer be assumed zero (or even constant) in the neighborhoods of the two planets.

In addition to the condition (230), conditions (81) (i), (ii), (iii) must be satisfied at each junction point.

B. Transfer between Circular Orbits about Two Planets

In this section, we will apply the general theory of the last section to the problem of optimal transfer of a rocket from a given circular orbit about the planet of departure into a given circular orbit about the planet of arrival. It will be assumed that escape from and entry into circular orbits are achieved by single impulsive thrusts applied as explained in Section X and that no other thrusts are necessary.

During the time the rocket is in the vicinity of the planet of departure, the contribution of the sun's field to the coefficients $\partial f_j/\partial x_i$ appearing in the Euler equations (45) will be negligible by comparison with that due to the planet's field. Hence, if we neglect the sun's field whilst the rocket is in the hyperbolic escape orbit, the Euler equations have the solution (114). In these equations (λ, μ, ν) are the primer components along and perpendicular to the radius vector from the planet's center to the rocket, e is the eccentricity of the escape orbit and f is the angle through which the radius vector has rotated from its position when the rocket was at the apse.

An impulsive thrust is applied at the apse in the direction of motion and conditions (81) (ii), (iii) must be satisfied at this point. When $f \to 0$, it will be found from Eqs. (111), (112a) that

$$I \to -\frac{1}{(e+1)} \qquad J \to 0. \tag{231}$$

Hence, conditions (81) (iii), (ii) require that at the apse

$$\begin{aligned} \lambda &= A - \frac{C}{(e+1)^2} = 0, \\ \mu &= B(1+e) + \frac{D}{1+e} = 1, \\ \nu &= \frac{E}{1+e} = 0, \\ \lambda\xi + \mu\eta + \nu\zeta &= -\frac{\gamma^{1/2}}{l^{3/2}}(1+e)A + C = 0, \end{aligned} \tag{232}$$

Eqs. (118) having been employed to formulate the last of these equations. It now follows that

$$A = C = E = 0, \qquad D = 1 + e - (1+e)^2 B \tag{233}$$

and therefore Eqs. (114) reduce to

$$\begin{aligned} \lambda &= Be \sin f, \\ \mu &= B(1 + e\cos f) + \frac{1 + e - (1+e)^2 B}{1 + e\cos f}, \\ \nu &= \frac{F \sin f}{1 + e\cos f}. \end{aligned} \tag{234}$$

The components of the time derivative of the primer are also obtainable from Eqs. (118) in the form

$$\begin{aligned} \xi &= -\frac{\sqrt{(\gamma l)}}{r^2}\left\{\frac{1 + e - (1+e)^2 B}{1 + e\cos f} + B\right\}, \\ \eta &= \frac{\gamma^{1/2}}{l^{3/2}} e\{1 + e - (1+e)^2 B\}\sin f, \\ \zeta &= \frac{\gamma^{1/2}}{l^{3/2}}(e + \cos f)F. \end{aligned} \tag{235}$$

On the hyperbolic escape trajectory

$$\frac{l}{r} = 1 + e\cos f \tag{236}$$

and hence, when the rocket has receded to a great distance along the asymptote,

$$\cos f = -1/e \qquad \sin f = (e^2 - 1)^{1/2}/e. \tag{237}$$

It now follows that, upon entry into the interplanetary transfer ellipse,

$$\lambda = (e^2 - 1)^{1/2}B,$$

$$\mu = \frac{r}{l}\{(1 + e) - (1 + e)^2B\},$$

$$\nu = \frac{(e^2 - 1)^{1/2}}{l} rF, \tag{238}$$

$$\xi = 0,$$

$$\eta = \frac{\gamma^{1/2}}{l^{3/2}}(e^2 - 1)^{1/2}\{1 + e - (1 + e)^2B\},$$

$$\zeta = \frac{\gamma^{1/2}}{el^{3/2}}(e^2 - 1)F.$$

Equations (238) determine the primer **p** and its first prime derivative (relative to a nonrotating frame) upon entry into the interplanetary trajectory. The Euler equations (45) have now to be integrated along this trajectory from these initial conditions. Since these equations are of the second order, known initial values of the primer and its first derivative are sufficient to fix the appropriate solution required. However, Eq. (230) must be satisfied on the interplanetary trajectory. It follows that the required solution of the Euler equations is in the form taken by Eqs. (114, 118), when C has been put equal to zero. Thus, attaching primes to quantities which refer to the interplanetary transfer ellipse

$$\lambda' = A' \cos f' + B'e' \sin f',$$

$$\mu' = -A' \sin f' + B'(1 + e' \cos f') + \frac{D' - A' \sin f'}{1 + e' \cos f'},$$

$$\nu' = (1 + e' \cos f')^{-1}(E' \cos f' + F' \sin f'),$$

$$\xi' = \frac{\sqrt{\gamma' l'}}{r'^2}\left(\frac{A' \sin f' - D'}{1 + e' \cos f'} - B'\right), \tag{239}$$

$$\eta' = \frac{\gamma'^{1/2}}{l'^{3/2}}\{-A'(e' + \cos f') + D'e' \sin f'\},$$

$$\zeta' = \frac{\gamma'^{1/2}}{l'^{3/2}}\{-E' \sin f' + (e' + \cos f')F'\}.$$

These components of the primer and its derivative are to be taken in directions along and perpendicular to r', the radius vector from the sun to the rocket.

Equations (239) must be equivalent to Eqs. (238) at the commencement of the transfer ellipse. Resolving (ξ', η', ζ') in the directions of η, ζ respectively and equating these components of $D\mathbf{p}$, two equations are

obtained from which we may solve for B and F. By inspection it is clear that

$$B = \frac{1}{1 + e} + O(\epsilon) \qquad F = O(\epsilon), \tag{240}$$

where $\epsilon = \sqrt{\gamma' l^3/\gamma l'^3}$. ϵ will always be very small in practice and hence we shall take

$$B = \frac{1}{1 + e} \qquad F = 0, \tag{241}$$

approximately. The content of the last pair of Eqs. (238) is represented by Eqs. (241). The remaining Eqs. (238) now require that

$$\lambda = \sqrt{\frac{e - 1}{e + 1}} \qquad \mu = \nu = 0 \qquad \xi = 0, \tag{242}$$

i.e. that, upon entry into the transfer ellipse, the primer shall be a vector of magnitude $\sqrt{(e - 1)/(e + 1)}$ in the direction of the velocity of recession from the planet of departure and that $D\mathbf{p}$ shall be perpendicular to this direction. Similar conditions must be satisfied at the end of the transfer ellipse upon entry into the hyperbolic orbit of approach to the planet of arrival.

These eight conditions will be sufficient to determine A', B', D', E', F', the position of the planet of departure in its orbit at the time of departure, the position of the planet of arrival in its orbit at the time of arrival and the transfer ellipse (there are ∞^1 such ellipses connecting two given points).

If the planet of departure is disregarded and it is imagined that the rocket is initially moving alone in this planet's orbit about the sun, it can be transferred into the interplanetary ellipse by means of an impulse generating a velocity increment **v**, where **v** is the ultimate velocity of recession from the planet of departure in the actual case. This impulse will be in the direction of **v**. The conditions (242) determining the optimal interplanetary ellipse can now be interpreted as requiring that the primer **p** shall be a vector of magnitude $\sqrt{(e - 1)/(e + 1)}$ in the direction of this hypothetical impulse and that $D\mathbf{p}$ shall be orthogonal to **p**. This statement of the conditions (242) is identical with the statement of conditions (81) (ii) and (iii) applicable to an actual impulse, except that the primer is no longer a *unit* vector. The following rule accordingly summarizes the results of this Section.

For the purpose of calculating the optimal interplanetary ellipse for a transfer between circular orbits about two planets, the planets may be disregarded and the problem solved as for the case of a two-impulse transfer between the planetary orbits themselves, provided that the requirement (81) (iii) is replaced by, "the primer must be a vector of magnitude $\sqrt{(e - 1)/(e + 1)}$ *in*

the direction of the thrust." In this context, e is the eccentricity of the hyperbolic trajectory of escape from the planet of departure or of approach to the planet of arrival.

This rule implies that, if the planetary orbits are coplanar, the analysis contained in Section VII is applicable to the present problem provided $\sin\phi$, $\cos\phi$ in Eqs. (119) are replaced by

$$\sqrt{\frac{e-1}{e+1}}\sin\phi \qquad \sqrt{\frac{e-1}{e+1}}\cos\phi$$

respectively. The effect on the conditions (134)–(136) is to multiply each left-hand member by

$$\sqrt{\frac{e_1-1}{e_1+1}}$$

and each right-hand member by

$$\sqrt{\frac{e_2-1}{e_2+1}},$$

e_1, e_2 being the eccentricities of the hyperbolae of escape and approach respectively. The conditions (139)–(146) are unaffected. These eleven conditions, after this modification, determine the optimal interplanetary ellipse very nearly as before.

However, equations for e_1, e_2 have yet to be given. These are obtained thus: Let v_1 be the ultimate velocity of recession from the planet of departure making an angle ϕ_1 with the perpendicular to the radius vector from the sun. Then, by Eq. (151),

$$v_1 = \gamma^{1/2}s_1(p^{-1/2} - p_1^{-1/2})\sec\phi_1, \tag{243}$$

γ, p, p_1, s_1 having the significance explained in Section VII (we are here assuming that the escape maneuver takes place in a region of negligible proportions). Also

$$v_1 = v_{0d}(e_1 - 1)^{1/2}, \tag{244}$$

where v_{0d} is the velocity in the circular orbit. Hence

$$v_{0d}(e_1 - 1)^{1/2} = \gamma^{1/2}s_1(p^{-1/2} - p_1^{-1/2})\sec\phi_1. \tag{245}$$

This equation determines e_1.

Similarly, it may be proved that

$$v_{0a}(e_2 - 1)^{1/2} = \gamma^{1/2}s_2(p_2^{-1/2} - p^{-1/2})\sec\phi_2, \tag{246}$$

ϕ_2 being the angle made by the velocity of approach to the planet of arrival with the perpendicular to the radius vector from the sun, and v_{0a} the velocity in the circular orbit about this planet. Equation (246) determines e_2.

The characteristic velocity of the whole transfer maneuver is given by

$$V = \sqrt{2v_{0d}^2 + v_1^2} - v_{0d} + \sqrt{2v_{0a}^2 + v_2^2} - v_{0a}, \tag{247}$$

v_2 being the velocity of approach to the planet of arrival from infinity.

The above results were first obtained in reference [20].

C. Transfer between the Surfaces of Two Planets

If it is required to effect transference between two points, one on the surface of each of two planets, a reasonable approximation to the optimal interplanetary ellipse can be found by neglecting the planetary atmospheres (if any) and rotations. The optimal mode of escape from the planet of departure is then via a hyperbolic orbit with its apse at the launching point. This is proved in reference [21]. To launch the rocket into the hyperbolic escape orbit, it is assumed that a horizontal impulse is applied at the point of departure. In practice, of course, the thrust will be finite and the rocket will be controlled, first to rise vertically until the atmospheric film has been penetrated and then to heel over until its track is horizontal. However, the idealized maneuver should permit the calculation of the required interplanetary ellipse with sufficient accuracy.

The approach maneuver is similarly idealized and then the motions along the hyperbolae of escape and approach are allowed for exactly as explained in Section XI,B. The optimal interplanetary ellipse is therefore identical with that appropriate to a transfer between circular orbits lying in the surfaces of the two planets. However, since the initial velocity of the rocket relative to the planet of departure and its final velocity relative to the planet of arrival are both zero, Eq. (247) for the characteristic velocity must be replaced by

$$V = \sqrt{2v_{0d}^2 + v_1^2} + \sqrt{2v_{0a}^2 + v_2^2}, \tag{248}$$

v_{0d}, v_{0a} now being the circular velocities at the points of departure and arrival respectively.

XII. Optimal Transfer between Planets in Given Positions

The optimal positions for the planets of departure and arrival in their orbits at the commencement of a transfer maneuver can be determined by the method described in Section XI,B. This optimal maneuver provides a standard with which all other maneuvers may be compared. However, its practical utility will be limited, since the required configuration of the two planets at the commencement of the maneuver will arise infrequently and, to avoid long delays, it will be necessary to be prepared to effect

transfer when the configuration is nonoptimal. A possible approach to this problem will now be outlined:

For a given pair of planets, assume the planet of departure to be at a certain position on its orbit at the instant of departure and the planet of arrival to be at a certain position on its orbit at the instant of arrival. The plane of the transfer ellipse is then known, but there is a single infinity of such ellipses passing through the assumed positions of the planets. By differentiation with respect to the parameter of this family, that transfer ellipse which corresponds to minimal propellant expenditure can be found. For the purposes of this calculation, it should be sufficient to regard the whole maneuver as comprising three phases, during each of which the rocket is subjected to the attraction of a single body. Having discovered a suitable transfer ellipse with the aid of this approximation, it will be necessary to correct the trajectory in the vicinity of the two planets to allow for the perturbation caused by the nonuniformity of the sun's field (see Section XIII). A transfer trajectory appropriate to given positions of the one planet upon departure and of the other upon arrival will thus become available. The time of transit can now be computed and then the position of the planet of arrival at the instant of departure is known. In this way, a transfer maneuver which may be satisfactorily employed when the two planets occupy a particular pair of positions at the commencement of the maneuver has been determined.

This calculation must now be repeated for a large number of different pairs of positions of the two planets and the results exhibited in a double entry table from which the elements of the transfer trajectory corresponding to given longitudes of the two planets in their orbits at the instant of departure may be read. By computing such a table for each pair of planets, the computation of transfer maneuvers for given planetary configurations at departure, would be greatly simplified.

A preliminary investigation of this type of problem will be found in reference [22].

XIII. Correction for Nonuniformity of the Sun's Field

In Section XI, the assumption has been made that the transfer maneuver comprises three phases during each of which the rocket's motion may be computed by supposing it subjected to the attraction of one body only. Over the major portion of the interplanetary ellipse, the fields of the planets of departure and arrival will be quite negligible by comparison with that of the sun. In the vicinity of either of the two planets, the sun's field will not be negligible, but it can, nonetheless, be neglected to a good degree of accuracy provided the rocket's motion is referred to a non-

rotating frame moving with the planet under consideration. For such a frame will be falling freely in the sun's field and, were this field uniform, its presence would not be detectable from observations with instruments stationary within the frame. Any error which arises as a result of our assumption will accordingly be due to the non-uniformity of the sun's field. This is small and its effect will be found now.

Let 0 be the center of one of the planets and $0xyz$ a nonrotating frame moving with this planet. Let (F, G, H) be the components of the sun's attraction at 0 and hence the components of 0's acceleration relative to an inertial frame. Let P be a particle moving in the vicinity of the planet and having coordinates (x, y, z). Let (f, g, h) be the components of the planet's attraction at P and $(F + \delta F, G + \delta G, H + \delta H)$ those of the sun's field. Then, relative to an inertial frame, P has acceleration components

$$\ddot{x} + F \qquad \ddot{y} + G \qquad \ddot{z} + H$$

and its equations of motion are accordingly

$$\ddot{x} + F = f + F + \delta F \qquad \text{etc.}$$

or

$$\ddot{x} = f + \delta F \qquad \text{etc.} \tag{249}$$

If the sun's field were uniform, $\delta F = \delta G = \delta H = 0$ and the motion of P relative to $0xyz$ would clearly be unaffected by the presence of this body. Its nonuniformity is revealed by the additional perturbing field $(\delta F, \delta G, \delta H)$.

Let 0 have coordinates (X, Y, Z) relative to a parallel frame with its origin at the center of the sun. Then, provided the distance 0P is small by comparison with the sun's distance from 0, we have very nearly

$$\delta F = \frac{\partial F}{\partial X} x + \frac{\partial F}{\partial Y} y + \frac{\partial F}{\partial Z} z \qquad \text{etc.} \tag{250}$$

But

$$F = -\frac{\gamma_S X}{(X^2 + Y^2 + Z^2)^{3/2}} \qquad \text{etc.} \tag{251}$$

and hence

$$\delta F = \frac{\gamma_S}{R^5} \{(2X^2 - Y^2 - Z^2)x + 3XYy + 3XZz\} \qquad \text{etc.} \tag{252}$$

where R is the sun's distance from 0. If now the directions of the axes at the instant under consideration are so chosen that $0x$ lies along the radius from the sun to the planet, $0xy$ is the plane of the planet's orbit and $0z$ is perpendicular to this plane, then $X = R$, $Y = Z = 0$ and

$$\delta F = \frac{2\gamma_S}{R^3} x \qquad \delta G = -\frac{\gamma_S}{R^3} y \qquad \delta H = -\frac{\gamma_S}{R^3} z. \tag{253}$$

Assuming that the planet is moving in a circular orbit with angular velocity ω,

$$\omega^2 = \gamma_S/R^3 \tag{254}$$

and Eqs. (253) may be written

$$\delta F = 2\omega^2 x \qquad \delta G = -\omega^2 y \qquad \delta H = -\omega^2 z. \tag{255}$$

In the case of the earth, $\omega = 1.991 \times 10^{-7}$ sec^{-1} and at a distance of 10^6 km this perturbation is of the order of 10% of the earth's gravitational acceleration which, in turn, is about 7% of the sun's field.

In order to maintain the axes $0xyz$ in the directions assumed above, it will be necessary to rotate them with an angular velocity ω about $0z$. To allow for this rotation, centrifugal and Coriolis forces must be introduced in addition to the perturbations δF, etc. The centrifugal acceleration has components $(\omega^2 x, \omega^2 y, 0)$ and hence the net perturbing acceleration (not including the Coriolis effect) has components

$$3\omega^2 x \qquad 0 \qquad -\omega^2 z. \tag{256}$$

The Coriolis acceleration is of magnitude $2\omega v$, where v is the component of P's velocity perpendicular to the z-axis, and is in a direction perpendicular to this velocity and the z-axis in a sense opposed to that of the rotation.

If (r, θ) are polar coordinates in the xy-plane and $\gamma/0P^2$ represents the attraction of the planet, the equations of motion of a body moving in the vicinity of this body can be written

$$\ddot{r} - r\dot{\theta}^2 = -\gamma r(r^2 + z^2)^{-3/2} + 2\omega r\dot{\theta} + 3\omega^2 r \cos^2\theta, \tag{257}$$

$$\frac{d}{dt}(r^2\dot{\theta}) = -2\omega r\dot{r} - 3\omega^2 r^2 \sin\theta \cos\theta, \tag{258}$$

$$\ddot{z} = -\gamma z(r^2 + z^2)^{-3/2} - \omega^2 z. \tag{259}$$

These equations will determine the motion of the rocket in the vicinity of the planet to a high degree of accuracy. However, their solution is a difficult matter and we shall be content with obtaining a rough assessment of the effect upon the trajectory of the perturbing forces. This is obtained as follows:

During most of the escape maneuver, the rocket will be moving approximately along an asymptote of the hyperbolic trajectory with a speed roughly equal to its final velocity of escape. Thus, relative to *nonrotating* axes $0xyz$ whose directions are chosen at the commencement of the maneuver as described above, the motion of the rocket is given very approximately by

$$x = ut\cos\alpha \qquad y = ut\sin\alpha \qquad z = vt, \tag{260}$$

where (u, v) are the components of the velocity of escape in the xy-plane and along $0z$ respectively and α is the angle made by the direction of the component u with $0x$. By Eqs. (255), the perturbing force therefore has approximate components

$$\delta F = 2\omega^2 ut\cos\alpha \qquad \delta G = -\omega^2 ut\sin\alpha \qquad \delta H = -\omega^2 vt. \tag{261}$$

These are very small and hence the crude approximation employed should, nonetheless, be adequate. These perturbing forces modify the uniform motion assumed in Eqs. (260) and result in velocity increments

$$\omega^2 ut^2\cos\alpha \qquad -\tfrac{1}{2}\omega^2 ut^2\sin\alpha \qquad -\tfrac{1}{2}\omega^2 vt^2, \tag{262}$$

being applied to the rocket at time t in the directions of the axes and additional displacements in these directions of

$$\tfrac{1}{3}\omega^2 ut^3\cos\alpha \qquad -\tfrac{1}{6}\omega^2 ut^3\sin\alpha \qquad -\tfrac{1}{6}\omega^2 vt^3. \tag{263}$$

Resolving these corrections in the directions of (i) the radius vector $\mathbf{r}$, (ii) the perpendicular to $\mathbf{r}$ in the xy-plane and (iii) $0z$, it will be found that the velocity corrections are

$$\begin{aligned} &\text{(i)} \ \tfrac{1}{2}\omega^2 ut^2(3\cos^2\alpha - 1), \\ &\text{(ii)} \ -\tfrac{3}{2}\omega^2 ut^2\sin\alpha\cos\alpha, \\ &\text{(iii)} \ -\tfrac{1}{2}\omega^2 vt^2, \end{aligned} \tag{264}$$

and the displacement corrections are

$$\begin{aligned} &\text{(i)} \ \tfrac{1}{6}\omega^2 ut^3(3\cos^2\alpha - 1), \\ &\text{(ii)} \ -\tfrac{1}{2}\omega^2 ut^3\sin\alpha\cos\alpha, \\ &\text{(iii)} \ -\tfrac{1}{6}\omega^2 vt^3. \end{aligned} \tag{265}$$

When applying these corrections to the hyperbolic motion, it seems reasonable to take t to be the time which would have elapsed if the rocket had been in rectilinear motion with the final escape velocity from the center of the planet to the point on the escape trajectory being considered.

Consider the case of a ship escaping from the earth along a trajectory lying in the plane of the planet's orbit and whose asymptote is perpendicular to the radius vector from the sun to the earth at the commencement of the maneuver. If the final velocity of escape is 3 km sec^{-1} (appropriate to a Martian transfer), then

$$\omega = 1.991 \times 10^{-7}\ \text{sec}^{-1} \qquad \alpha = \tfrac{1}{2}\pi \qquad u = 3{,}000\ \text{m sec}^{-1}, \qquad v = 0.$$

When the ship has receded to a distance of 10^6 km from the earth's center, $t = 10^6/3$ sec, and the corrections to the velocity are calculated to be

(i) 0.007 km sec^{-1} (ii) 0 (iii) 0.

The displacement corrections are

(i) -730 km (ii) 0 (iii) 0.

As has been remarked above, at this distance the earth's field is only 7% of the sun's field and the later portion of the trajectory can be found by neglecting the planetary fields until the planet of arrival is approached. The above figures suggest that the perturbing influence of the sun during the escape and approach maneuvers may be neglected.

XIV. General Theory of Correctional Maneuvers

When, subsequent to an impulsive thrust applied at a junction point, the rocket is found to diverge from the previously calculated optimal trajectory, the whole thrust program for the maneuver should, in theory, be recalculated. However, if the divergence is small, this recalculated program will not differ greatly from the original program and it seems reasonable to expect that, by adhering to this original program in spite of the divergence, the extra propellant expenditure involved will be negligible. This being the case, an instant t_0 will be selected at which it is proposed to execute the correction. The position P and velocity of the rocket on its actual path at this instant can then be calculated. If t_1 is the required instant of arrival at the next junction point J, the time of transit from P to J must be $t_1 - t_0$. In a given gravitational field, there will be but one arc of null-thrust passing through P and J along which the time of transit is $t_1 - t_0$. The correctional thrust applied at P must be such as to transfer the rocket into this arc.

Thus, let (x_0, y_0, z_0) be the Cartesian coordinates of P. These coordinates can be computed when the elements of the erroneous orbit are known. Let (x_1, y_1, z_1) be the coordinates of J. The equations of motion of a body moving in a given gravitational field are three in number and each is of the second order. Their solution accordingly involves six constants of integration. Consider the solution corresponding to a body leaving the point P at time t_0 with velocity components (u_0, v_0, w_0). This may be written,

$$\begin{aligned} x &= x(t, x_0, y_0, z_0, u_0, v_0, w_0), \\ y &= y(t, x_0, y_0, z_0, u_0, v_0, w_0), \\ z &= z(t, x_0, y_0, z_0, u_0, v_0, w_0), \end{aligned} \tag{266}$$

x_0, etc., determining the integration constants. The velocity components (u_0, v_0, w_0) have to be selected so that when $t = t_1$, then $x = x_1$, $y = y_1$ and $z = z_1$, that is, so that

$$\begin{aligned} x_1 &= x(t_1, x_0, y_0, z_0, u_0, v_0, w_0), \\ y_1 &= y(t_1, x_0, y_0, z_0, u_0, v_0, w_0), \\ z_1 &= z(t_1, x_0, y_0, z_0, u_0, v_0, w_0). \end{aligned} \tag{267}$$

This set of equations constitutes three equations for the three unknowns (u_0, v_0, w_0) and hence, there will be a unique orbit joining P and J and satisfying the required conditions. There may, of course, be a number of distinct solutions of the Eqs. (267), but only one will lie wholly in the neighborhood of the optimal track and can therefore be followed with little extra expenditure of propellant. At any point P, therefore, the necessary correcting action is uniquely determined.

Since the required trajectory PJ will not diverge greatly, either from the optimal track, or from the trajectory the rocket was actually following prior to its arrival at P, the solution of Eqs. (267) may be assumed to be of the form

$$u_0 = u + \delta u \qquad v_0 = v + \delta v \qquad w_0 = w + \delta w, \tag{268}$$

where (u, v, w) are the velocity components with which the rocket arrives at P and $(\delta u, \delta v, \delta w)$ are the components of the velocity increment necessary to effect correction and are small. Substituting in Eqs. (267) and neglecting second order terms, we obtain

$$\begin{aligned} x_1 - x_1' &= \frac{\partial x_1'}{\partial u}\delta u + \frac{\partial x_1'}{\partial v}\delta v + \frac{\partial x_1'}{\partial w}\delta w, \\ y_1 - y_1' &= \frac{\partial y_1'}{\partial u}\delta u + \frac{\partial y_1'}{\partial v}\delta v + \frac{\partial y_1'}{\partial w}\delta w, \\ z_1 - z_1' &= \frac{\partial z_1'}{\partial u}\delta u + \frac{\partial z_1'}{\partial v}\delta v + \frac{\partial z_1'}{\partial w}\delta w, \end{aligned} \tag{269}$$

where $x_1' = x(t_1, x_0, y_0, z_0, u, v, w)$, etc. (x_1', y_1', z_1') is the point J′ at which the rocket will arrive at time t_1 if no correction is made. Solving Eqs. (269) for δu, δv, δw, we obtain

$$\frac{\partial(x_1', y_1', z_1')}{\partial(u, v, w)}\delta u = (x_1 - x_1')\frac{\partial(y_1', z_1')}{\partial(v, w)} + (y_1 - y_1')\frac{\partial(z_1', x_1')}{\partial(v, w)} + (z_1 - z_1')\frac{\partial(z_1', y_1')}{\partial(v, w)} \quad \text{etc.} \tag{270}$$

These equations determine the velocity correction necessary at P.

The arguments of the coefficients $\partial(y_1', z_1')/\partial(v, w)$, etc. in Eqs. (270) are $(t_1, x_0, y_0, z_0, u, v, w)$. These coefficients have therefore to be calculated at the instant t_1 on the erroneous orbit through P. This calculation cannot be performed, therefore, until the elements of this orbit are available. To avoid deferring the computation of these coefficients to such a late juncture,

it will be advisable to approximate to their values beforehand by finding the values they assume on the programmed track at various instants $t = t_0$, i.e. by replacing (u, v, w) by (u_0, v_0, w_0). Employing these values in the Eqs. (270), approximate values for the components of the velocity correction will become available in a few minutes. The fact that accurate values of δu, δv, δw are not used in the correction maneuver, is of no great importance, since (a) it will not be easy to apply any correction demanded with great precision, and (b) any inaccuracy can be allowed for later when corrections again become necessary. This method of correction is illustrated in reference [23].

XV. Perturbation Maneuvers

Very little investigation has been carried out relating to these maneuvers. The principle is explained in reference [24]. The general theory of optimal trajectories as developed in Section IV embraces such a possibility. The problem is to optimize the position of the attracting body responsible for the perturbation at the instant of departure. This position on the body's track can be specified in terms of a single parameter and the given gravitational field structure will then involve this unknown quantity. If it be denoted by κ, the theory of Section IV shows how its optimal value may be obtained [Eq. (65)].

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