Interplanetary Trajectories with Excess Energy

By

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(With 14 Figures)

Abstract — Zusammenfassung — Résumé

Interplanetary Trajectories with Excess Energy. Several families of interplanetary trajectories have been analyzed in an effort to reduce total round-trip time with the least excess energy. Velocity increments Δv were calculated on the basis of a round trip from a circular orbit near the Earth to a circular orbit near the destination. On this basis, the round trip to Venus was reduced from the least-energy value of 760 days with a total Δv of 8.22 miles per second to a round-trip time of 400 days with a total Δv of 9.7 miles per second. For the Mars journey, the round-trip time was reduced from the minimum-energy value of 973 days with total Δv of 6.96 miles per second to a round-trip time of 400 days with a total Δv of 14.9 miles per second. These reductions, which were the best found in the range of moderate Δv, correspond to zero waiting time at the destination planets. If the waiting time is increased moderately, or if further reductions in travel time are desired, considerably larger total velocity increments must be employed. Reduction in trip time to about 180 days requires a total Δv of 13.6 miles per second for the Venus trip, and about 29 miles per second for the Mars trip. In general, it is concluded that reductions in round-trip time to Venus are achievable with much smaller velocity increments than are similar reductions in round-trip time to Mars.

Interplanetarische Reiserouten mit Überschußenergie. Es wurden einige Familien interplanetarischer Reiserouten in dem Bemühen analysiert, die Gesamtzeit für die Rundreise mit geringstem Energieverbrauch zu reduzieren. Die Geschwindigkeitsinkremente Δv wurden auf der Grundlage einer Rundreise berechnet, die ihren Anfang von einer erdnahen Kreisbahn nimmt und zu einer Kreisbahn in der Nähe des Zielplaneten führt. Mit solchen Annahmen wurde die Rundreisezeit zur Venus von 760 Tagen bei geringstem Energieverbrauch mit einem Gesamt-Δv von 8,22 Meilen/sec auf eine Rundreisezeit von 400 Tagen mit einem Gesamt-Δv von 9,7 Meilen/sec verkürzt. Für die Marsreise wurde die Rundreisezeit von 973 Tagen bei geringstem Energieaufwand mit einem Gesamt-Δv von 6,96 Meilen/sec auf eine Rundreisezeit von 400 Tagen mit einem Gesamt-Δv von 14,9 Meilen/sec verkürzt. Diese Verkürzungen, welche die besten Ergebnisse im Bereich von mäßigem Δv waren, entsprechen einer Wartezeit von null auf den Zielplaneten. Wenn die Wartezeit mäßig erhöht wird, oder wenn weitere Verringerungen der Reisezeit erwünscht sind, müssen beträchtlich größere Gesamtgeschwindigkeitsinkremente benützt werden. Die Verkürzung der Reisezeit auf ungefähr 180 Tage erfordert ein Gesamt-Δv von 13,6 Meilen/sec für die Reise zur Venus und von ungefähr 29 Meilen/sec für die Fahrt zum Mars. Im allgemeinen ergibt sich die Schlußfolgerung, daß Verringerungen der Rundreisezeit zur Venus mit viel kleineren Geschwindigkeitsinkrementen erreichbar sind als ähnliche Verkürzungen der Rundreisezeit zum Mars.

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Trajectoires interplanétaires avec excédent d'énergie. Plusieurs familles de trajectoires interplanétaires ont été analysées dans le but de réduire la durée totale du voyage avec un minimum d'excédent d'énergie. Les accroissements de vitesse Δv ont été calculés pour un aller-retour entre deux orbites circulaires, l'une autour de la terre, l'autre autour de la planète de destination. Dans le cas de Venus les 760 jours et 8.22 miles par seconds de Δv, correspondant à la dépense énergétique minimum, ont été réduits à 400 jours pour 9.7 miles par seconde. Dans le cas de Mars on peut passer de 973 jours et 6.96 miles par seconde à 400 jours et 14.9 miles par seconde. Ces réductions, les plus grandes trouvées dans la zone des Δv modérés, correspondent à un temps d'attente nul aux planètes de destination. S'il existe un temps d'attente modéré, ou si l'on veut réduire la durée davantage, il faut des accroissements de vitesse beaucoup plus considérables. Pour Venus la réduction de la durée du voyage à 180 jours demande 13.6 miles par seconde et 29 miles par seconde dans le cas de Mars. En conclusion, les réductions de temps pour Venus sont réalisables avec des accroissements de vitesse beaucoup plus modérés que pour Mars.

Introduction

One of the discouraging features of manned interplanetary travel is the length of time required for the journeys if minimum-energy transfer orbits are followed. A large portion of this time, for journeys to the near planets, is spent at the destination planet waiting for the Earth and the planet to move into the proper relative position so that the spaceship and the Earth will arrive at the same point at about the same time. Because of this conjugation problem, simple reductions in transit time do not necessarily produce reductions in total round-trip time. Such reductions may increase the waiting time by an amount equal to or even greater than the saving in transit time. To determine how total trip time can be reduced with the least excess energy requires systematic calculation of the time-components of the trip as function of the velocity increments for several promising families of trajectories. Results of such an investigation are presented herein[1].

Symbols

A	$\pm\sqrt{\varrho\,(V_0^2 - 2) + 2\varrho - V_0^2}$, eqs. (21) and (23)
E	total energy per unit mass
G	universal gravitational constant, 7.28×10^{-21}, mile^3/lb-sec^2
h	angular momentum per unit mass
M	mass of body producing gravity field
r	distance from center of body producing gravity field
r_0	reference distance
R_0	surface radius of planet
t	time
V	nondimensional velocity $v/v_{c,0}$,
V_0	nondimensional velocity at apsis of trajectory
v	velocity
v_c	circular velocity, $\sqrt{\mu/r}$
$v_{c,0}$	reference circular velocity, $\sqrt{\mu/r_0}$
v_h	hyperbolic velocity
v_0	velocity at apsis of trajectory
α	local angle of trajectory relative to circumferential direction
ε	eccentricity of trajectory
θ	angular distance from apsis
$\dot{\theta}$	angular velocity
θ_T	angular distance along trajectory between orbits of origin and destination
μ	gravitational constant for field-producing body, GM
ϱ	nondimensional distance from center of body producing gravity field, r/r_0
τ	nondimensional time, $(v_{c,0}/r_0)t$

[1] Another analysis of interplanetary orbits as function of energy was presented recently in [1]. This analysis, although very comprehensive and elegant for the one-way trip, did not consider the roundtrip, or conjugation problem. It differs also from the present analysis in that results were obtained in terms of heliocentric parameters, so that considerably more work is required to determine velocity increments required at satellite stations or at planet surfaces.

Trajectory Equations

The trajectories to be considered are all segments of conic sections with the sun at one focus. This implies that third- or fourth-body interaction effects are neglected, and that no thrust is applied except during brief periods at the beginning and end of the trajectory.

The equation for the conic-section trajectory followed in a central gravitational field is (see, e.g., [2])

$$r = \frac{h^2}{\mu}(1+\varepsilon\cos\theta)^{-1} \tag{1}$$

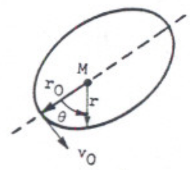

Fig. 1

where the eccentricity ε is

$$\varepsilon = \sqrt{1+\frac{2Eh^2}{\mu^2}}. \tag{2}$$

The angle θ (Fig. 1) is measured from the apsis closest to the gravitational mass M. For generality, it is convenient to nondimensionlize the trajectory equation in terms of the apsis distance r_0, and the circular velocity at r_0, $v_{c,0}$, where

$$v_{c,0} = \sqrt{\frac{\mu}{r_0}}. \tag{3}$$

If we denote r/r_0 by ϱ and $v/v_{c,0}$ by V, and note that

$$h = r_0 v_0 \tag{4}$$

and

$$E = \frac{v^2}{2} - \frac{\mu}{r} = \frac{v_0^2}{2} - \frac{\mu}{r_0} \tag{5}$$

the trajectory equation becomes

$$\varrho = V_0^2(1+\varepsilon\cos\theta)^{-1} \tag{6}$$

where

$$V_0 = \frac{v_0}{v_{c,0}} \tag{7}$$

$$\varepsilon = \sqrt{1+V_0^2(V_0^2-2)} = \pm(V_0^2-1). \tag{8}$$

It is evident from eq. (8) that $V_0^2 < 2$ corresponds to elliptic trajectories, $V_0^2 = 2$ to parabolic trajectories, and $V_0^2 > 2$ to hyperbolic trajectories.

It is convenient to keep only the plus sign in the expression for eccentricity, because eq. (6) then yields $\varrho = 1$ for $\theta = 0$; consequently, either apsis of an ellipse can be regarded as the reference point or origin. Thus, if V_0 is less than unity (velocity at r_0 less than the circular velocity), ϱ is maximum at $\theta = 0$; whereas if V_0 is greater than unity, ϱ is minimum at $\theta = 0$. The trajectory equation then becomes

$$\varrho = \frac{V_0^2}{1+(V_0^2-1)\cos\theta}. \tag{9}$$

The value of ϱ at $\theta = \pi$ is the maximum or minimum radius ratio, depending on whether V_0^2 is less than or greater than unity:

$$\varrho_m = \frac{V_0^2}{2-V_0^2}. \tag{10}$$

Trajectory Variables as Function of Time

In order to calculate excess-energy trajectories, it is necessary to know the coordinates ϱ and θ, the velocity ratio V, and the angle of the trajectory relative to the circumferential direction α, as functions of time. The velocity ratio V is obtained in terms of ϱ and V_0 by means of the energy eq. (5):

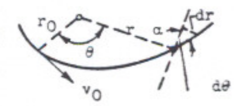

Fig. 2

$$V^2 = V_0^2 - 2\left(1 - \frac{1}{\varrho}\right). \tag{11}$$

The trajectory angle relative to the circumferential direction is obtained from the relation (see Fig. 2)

$$\alpha = \tan^{-1}\frac{dr}{r\,d\theta} = \tan^{-1}\frac{d\varrho}{\varrho\,d\theta}. \tag{12}$$

From eq. (9),

$$\frac{d\varrho}{\varrho\,d\theta} = \frac{(V_0^2-1)\sin\theta}{1+(V_0^2-1)\cos\theta} \tag{13}$$

$$\cos\theta = \frac{V_0^2-\varrho}{\varrho\,(V_0^2-1)} \tag{14}$$

$$\sin\theta = \frac{\pm V_0}{\varrho\,(V_0^2-1)}\sqrt{\varrho^2\,(V_0^2-2)+2\,\varrho - V_0^2}. \tag{15}$$

Consequently, in terms of V_0 and ϱ,

$$\frac{d\varrho}{\varrho\,d\theta} = \tan\alpha = \pm\frac{1}{V_0}\sqrt{\varrho^2\,(V_0^2-2)+2\,\varrho - V_0^2} \tag{16}$$

where the plus is used for $V_0 > 1$ and minus for $V_0 < 1$. The angular distance θ can be calculated from eq. (14). It is therefore necessary only to calculate ϱ as function of time for various values of V_0 in order to obtain all necessary trajectory variables as functions of time.

The variation of ϱ with time is obtained by integration of the equation expressing conservation of angular momentum:

$$r^2\frac{d\theta}{dt} = h = \text{constant} = r_0\,v_0. \tag{17}$$

To express time in convenient nondimensional form, let

$$\tau = \frac{v_{c,0}}{r_0}\,t. \tag{18}$$

Then eq. (17) becomes

$$d\tau = \frac{\varrho^2}{V_0}\,d\theta. \tag{19}$$

Substituting from eq. (16) for $d\theta$ in terms of $d\varrho$ gives

$$\tau = \pm\int_{\varrho_1}^{\varrho_2}\frac{\varrho\,d\varrho}{\sqrt{\varrho^2(V_0^2-2)+2\varrho-V_0^2}} \tag{20}$$

where the plus is used for $V_0^2 > 1$ ($\varrho_2 > \varrho_1$) and the minus for $V_0^2 < 1$ ($\varrho_2 < \varrho_1$).

The integrals of eq. (20), from $\varrho_1 = 1$ to $\varrho_2 = \varrho$, are as follows:

For $V^2 > 2$ (hyperbolic trajectories):

$$\tau = \frac{A}{V_0^2-2} - \frac{1}{(V_0^2-2)^{3/2}}\ln\left|\frac{A\sqrt{V_0^2-2}+(V_0^2-2)\varrho+1}{V_0^2-1}\right|. \tag{21}$$

For $V_0^2 = 2$ (parabolic trajectory):

$$\tau = \sqrt{2\,(\varrho-1)}\left(\frac{\varrho+2}{3}\right). \tag{22}$$

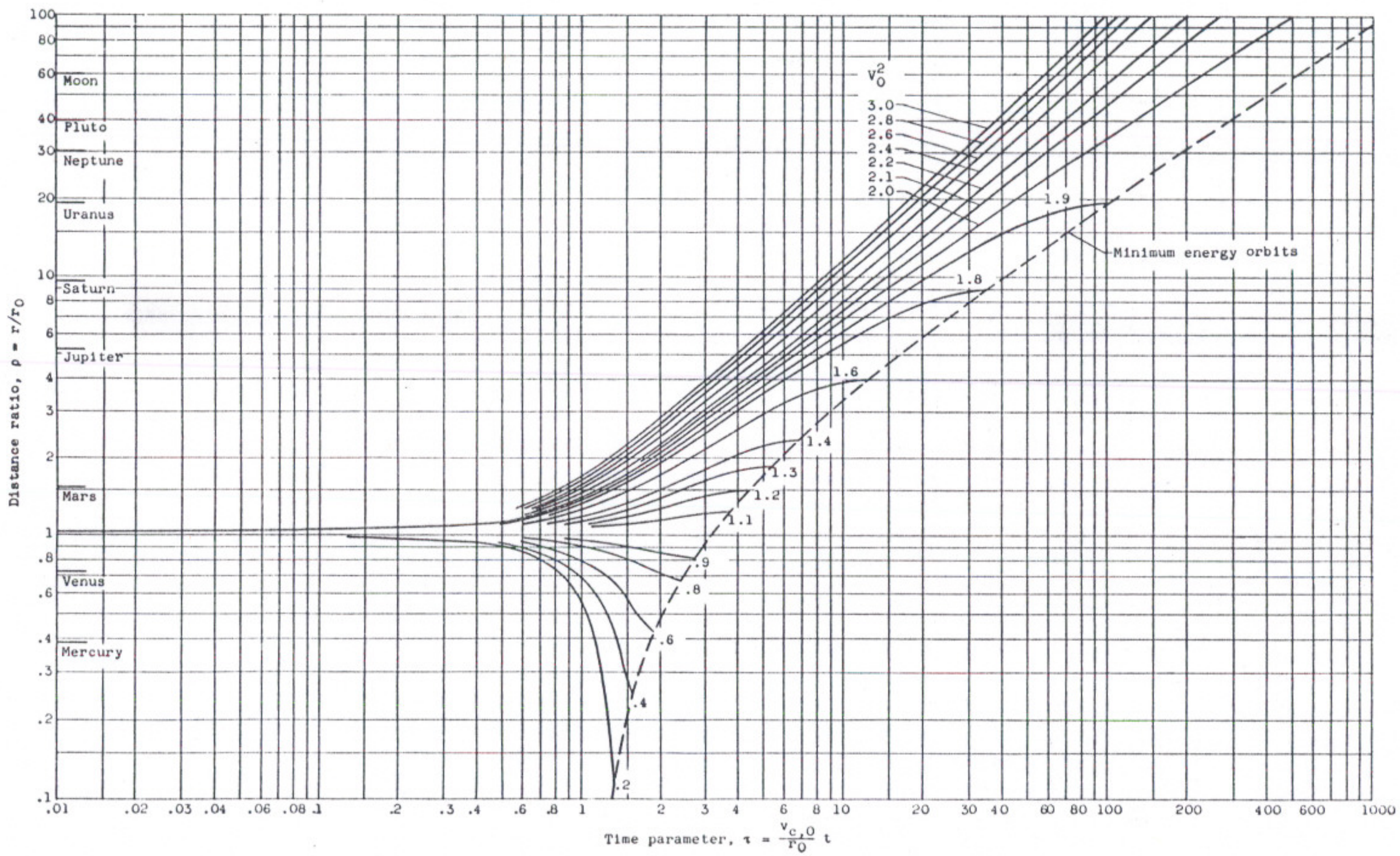

Fig. 3. Trajectories in a gravity field for several initial velocities, parallel to circular orbit at r_o

For $V_0^2 < 2$ (elliptic trajectories):

$$\tau = \frac{1}{(2-V_0^2)^{3/2}}\left[\sin^{-1}\frac{(2-V_0^2)\varrho-1}{V_0^2-1}+\frac{\pi}{2}\right]-\frac{A}{2-V_0^2} \tag{23}$$

where

$$A = \pm\sqrt{\varrho^2(V_0^2-2)+2\varrho-V_0^2}$$

in which the plus is used for $V_0^2 > 1$ and the minus for $V_0^2 < 1$.

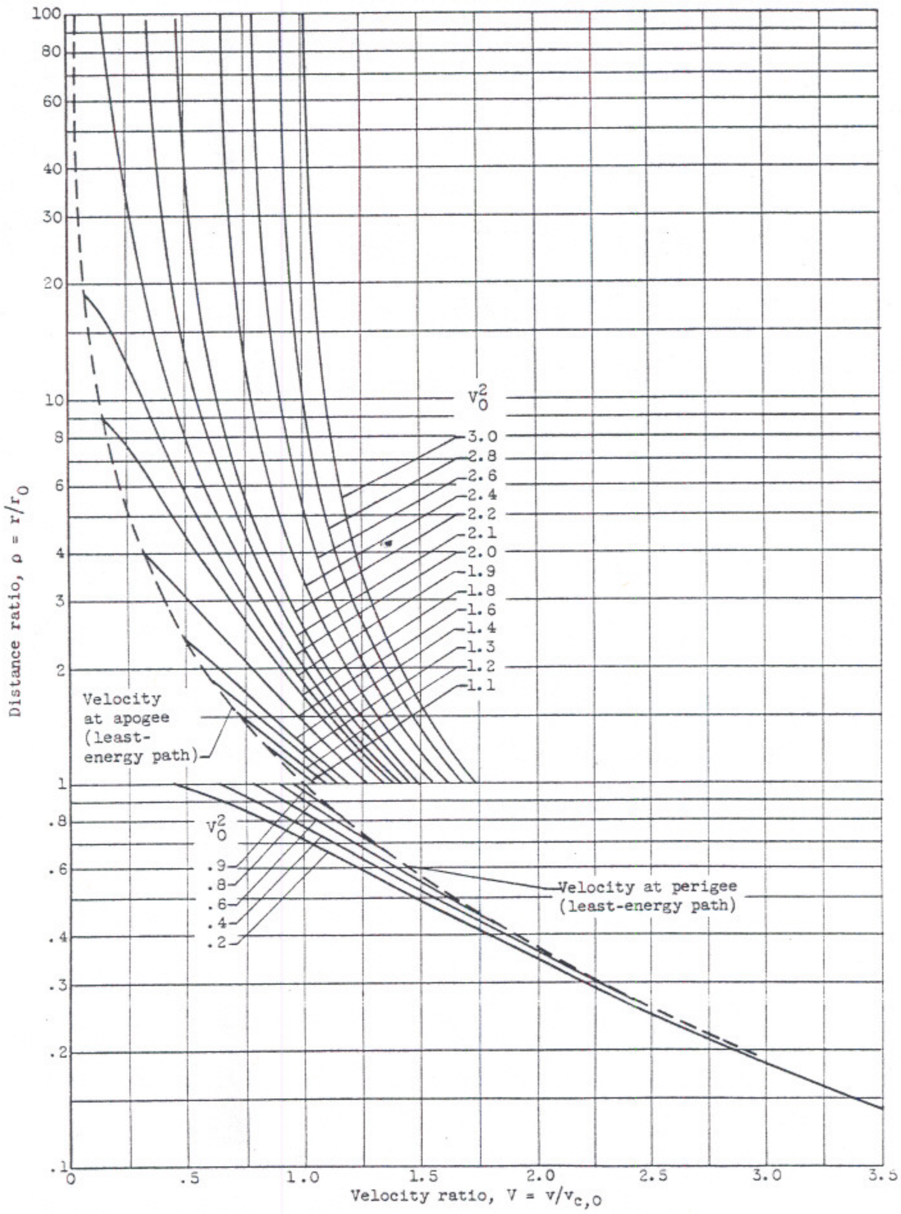

Fig. 4. Velocity along trajectory

The trajectories ϱ against τ are plotted in Fig. 3 for values of V_0^2 from 0.2 to 3.0. The local velocity ratio, the trajectory angle α, and the angular distance θ, are plotted as function of ϱ in Figs. 4, 5, and 6, respectively. These curves are general for any central gravitational field.

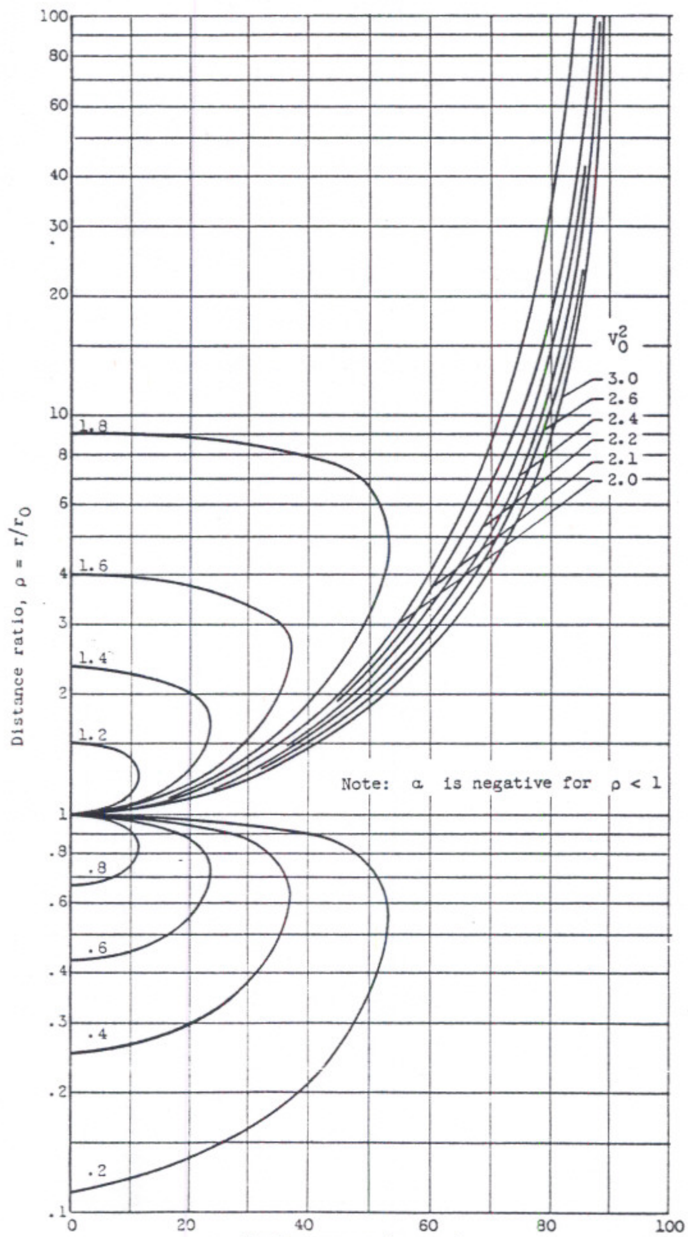

Fig. 5. Trajectory angle relative to circumferential direction

V_0^2 from 0.2
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Entering and Leaving Satellite Orbits

In order to provide a common basis for comparisons of excess-energy trajectories, the velocity impulses required will be referred to a standard mission, consisting of a round trip from a circular satellite orbit having a radius 1.1 times

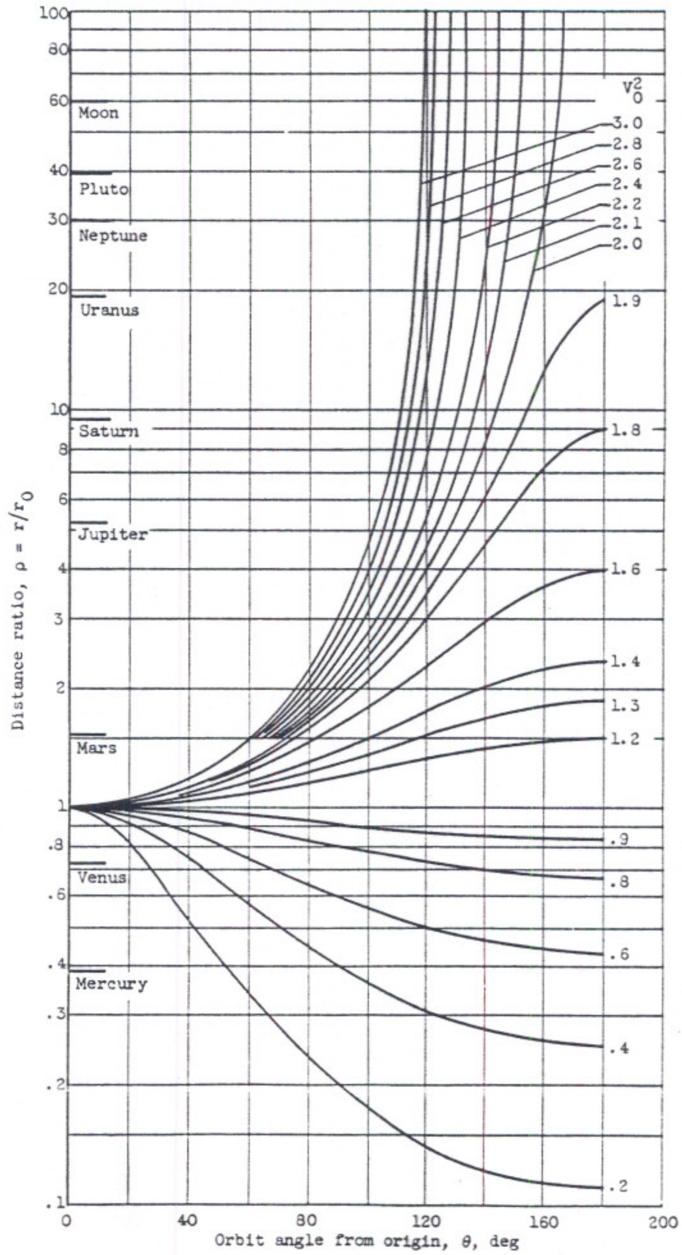

Fig. 6. Angular distance from origin of trajectory

Table I. *Data on*

Planet	Sun	Mercury	Venus	Earth
Symbol	☉	☿	♀	⊕
Mean distance from sun, miles	—	36×10^6	67.2×10^6	92.9×10^6
Mean orbital velocity, miles/sec	—	29.7	21.7	18.5
Mean angular orbital velocity, deg/day	—	4.09	1.602	0.986
Sideral period, days	—	88.0	224.7	365.26
Rotational period, days	24.65 to 33.3	88.0	30 (?)	0.998
Eccentricity of orbit	—	0.206	.00681	0.0167
Inclination of equator to orbit, deg	7.175	—	—	23.45
Inclination of orbit to ecliptic, deg	(to ecliptic) —	7.0	3.395	0
Mass, lb	4.35×10^{30}	0.725×10^{24}	10.65×10^{24}	13.19×10^{24}
Density (water = 1.00)	1.41	5.46	5.06	5.52
Force constant, $\mu = GM$, miles³/sec²	3.17×10^{10}	5.28×10^3	7.75×10^4	9.60×10^4
Solar radiation intensity (⊕ = 1)	—	6.7	1.9	1.0
Mean diameter, miles	864,000	3010	7610	7918
Oblateness $(a—b)/a$	0	0	0	0.00337
Gravity at surface, ft/sec²	897.0	12.3	28.2	32.2
Escape velocity at surface, miles/sec	383.0	2.65	6.38	6.97
Satellite velocity at surface, miles/sec	271.0	1.87	4.51	4.93
Sun's gravity at planet's orbit, ft/sec²	—	0.129	0.0371	0.0194

e I. *Data on* *the Solar System*

Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto	Moon (relative to Earth)
⊕	♂	♃	♄	♅	♆	♇	
92.9×10^6	141.5×10^6	483×10^6	886×10^6	1783×10^6	2791×10^6	3671×10^6	23.9×10^4
18.5	15.0	8.11	5.96	4.22	3.37	2.94	0.636
0.986	0.524	0.083	0.0335	0.0117	0.006	0.004	13.17
365.26	687.0	4332.0	10,759	30,700	60,200	90,800	27.32
0.998	1.027	0.413	0.431	0.448	0.666	—	27.32
0.0167	0.0933	0.0484	0.0558	0.0471	0.0086	0.249	0.0549
23.45	25.20	3.115	26.745	98.0	29.0	—	6.678
0	1.85	1.307	2.492	0.772	1.773	17.315	5.141
13.19×10^{24}	1.41×10^{24}	4150×10^{24}	1240×10^{24}	190×10^{24}	225×10^{24}	12.1×10^{24}	0.162×10^{24}
5.52	4.12	1.35	0.71	1.56	2.47	2	3.33
9.60×10^4	1.026×10^4	3.02×10^7	9.03×10^6	1.38×10^6	1.64×10^6	8.8×10^4	1.18×10^3
1.0	0.43	0.04	0.01	0.003	0.001	0.0006	1.0
7918	4140	86,900	71,500	29,500	26,800	3600.0	2160
0.00337	0.00521	0.065	0.1053	0.071	0.0222	—	—
32.2	12.7	84.4	36.9	33.5	48.2	14.3	5.33
6.97	3.15	37.3	22.4	13.7	15.7	10.0	1.48
4.93	2.23	26.4	15.8	9.68	11.1	7.0	1.05
0.0194	0.0084	0.00072	0.00021	0.000053	0.000021	0.000012	0.0194

the Earth's radius, to a circular satellite orbit at the destination planet, at a radius 1.1 times the planet's radius. We must, therefore, determine the relation between the heliocentric trajectories, which can be obtained from Figs. 3 to 6, and the velocity increments required in the satellite orbit.

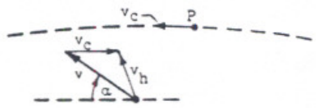

Fig. 7

The velocity increment required when a ship leaves or enters a satellite orbit is determined by the hyperbolic velocity of the ship; that is, by its velocity relative to the planet at large distances from the planet. The relation between hyperbolic velocity v_h, heliocentric approach or departure velocity, v, and the planet's velocity is shown in Fig. 7. The planet's velocity, of course, is close to the local circular velocity if the orbital eccentricity is small, as it is for most planets of interest. Assuming that the planet's velocity is exactly the circular velocity,

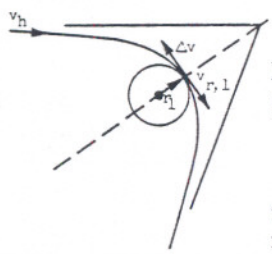

Fig. 8

$$v_h = \sqrt{(v \cos \alpha - v_c)^2 + (v \sin \alpha)^2}. \tag{24}$$

For transfer trajectories tangent to the planetary orbit ($\alpha = 0$), the hyperbolic velocity is simply

$$v_h = v - v_c. \tag{25}$$

When the hyperbolic velocity has been determined, the velocity increment required at the desired satellite radius r_1 is obtained from the energy equation. The energy per unit mass relative to the planet at large distances from the planet is $v_h^2/2$; and, since this remains constant, the velocity attained on the hyperbolic orbit at $r = r_1$ is (see Fig. 8)

$$v_{r,1}^2 = v_h^2 + 2 \frac{\mu}{r_1} = v_h^2 + 2\, v_{c,1}^2. \tag{26}$$

To settle into a circular orbit at r_1, the velocity there must be reduced to the circular velocity. Consequently, the required velocity impulse is

$$\Delta v = v_{r,1} - v_{c,1} = \sqrt{v_h^2 + 2\, v_{c,1}^2} - v_{c,1}. \tag{27}$$

With eqs. (27) and (24), the Δv required for any heliocentric approach or departure velocity and angle can be calculated. Since we are also interested in total trip time, we must now determine relations between transit time, waiting time, and the heliocentric trajectory variables.

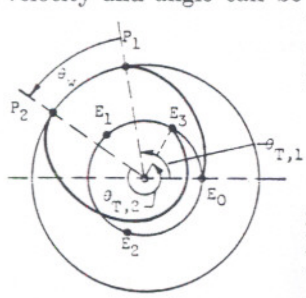

Fig. 9

Equations for Trip Time

The procedure for calculating trip times is illustrated in Fig. 9, where E and P refer to the location of the Earth and the destination planet at various times, θ_{T1} and θ_{T2} are the angular distance covered by the ship trajectory between the two orbits on the way to the planet and on the return trip, and θ_w is the angular movement of planet P during the waiting time t_w. A case is shown in which the journey to the planet is along the short leg of an ellipse, and the return trip is along the long leg of a different ellipse. In order that the spaceship and the Earth arrive at the same place at about the same time, we must have

$$\theta_{E,3} - 2n\pi = \theta_{T,1} + \theta_{T,2} + \theta_{w}. \tag{28}$$

Eq. (28) can be rewritten as

$$\dot{\theta}_E\,(t_1 + t_w + t_2) - 2\,n\,\pi = \theta_{T,1} + \theta_{T,2} + \dot{\theta}_P t_w \tag{29}$$

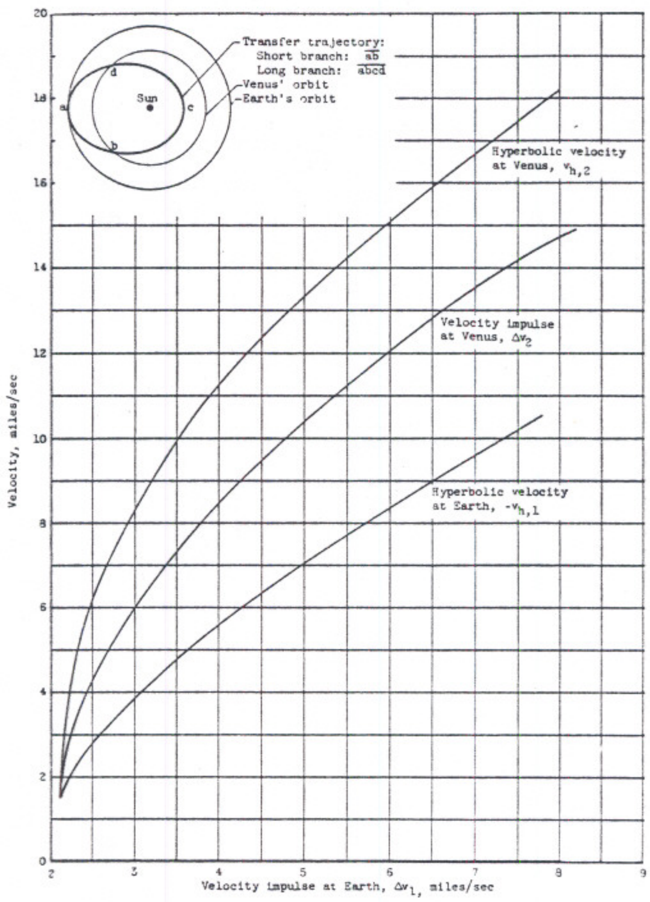

Fig. 10. Excess-energy paths between Earth and Venus, starting and ending in circular orbits at $r = 1.1\ R_{planet}$. *a*) Trajectory tangent at Earth's orbit: velocities

where $\dot{\theta}_E$ and $\dot{\theta}_P$ are the angular velocities of the Earth and the planet. Solving for t_w yields

$$t_w = \frac{\theta_{T,1} + \theta_{T,2} - \dot{\theta}_E\,(t_1 + t_2) + 2\,n\,\pi}{\dot{\theta}_E - \dot{\theta}_P}. \tag{30}$$

For a minimum-energy orbit, $\theta_{T,1} = \theta_{T,2} = \pi$, and eq. (30) becomes

$$t_{w,m.e.} = \frac{2\,(n+1)\,\pi - \dot{\theta}_E t_1}{\dot{\theta}_E - \dot{\theta}_P}. \tag{31}$$

In eqs. (30) and (31), the minimum value of n is used for which t_w is positive. For planets closer to the sun than the Earth ($\dot{\theta}_P > \dot{\theta}_E$), these formulas apply directly if negative values of n are used.

Another convenient form of eq. (29) is as follows:

$$\theta_{T,2} - \dot{\theta}_E t_2 = -(\theta_{T,1} - \dot{\theta}_E t_1) + (\dot{\theta}_E - \dot{\theta}_P) t_w - 2n\pi. \qquad (32)$$

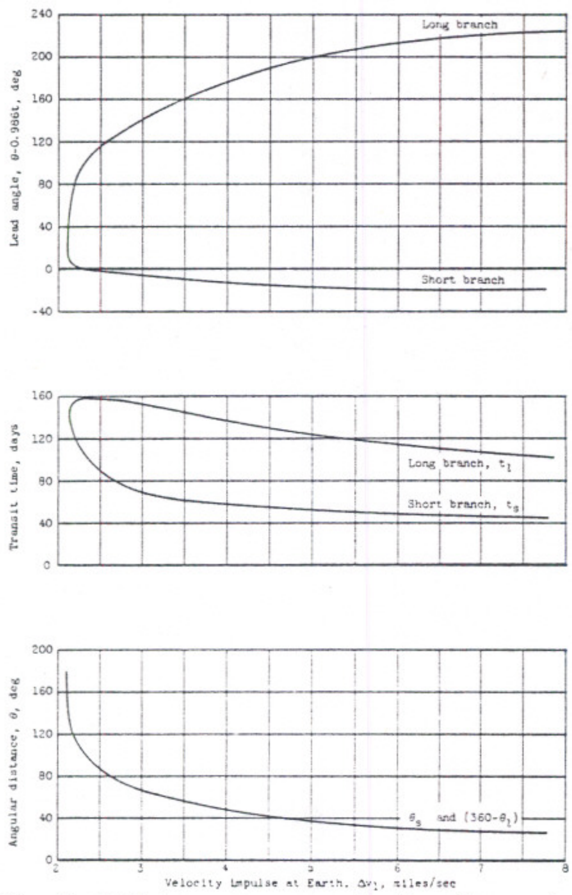

Fig. 10. *b*) Trajectory tangent at Earth's orbit: angular distance and time

The quantity $\theta_T - \dot{\theta}_E t$ is the lead angle acquired by the spaceship relative to the Earth in the trip between orbits. Eq. (32) shows the obvious result that, for $t_w = 0$ and $n = 0$, a lag angle acquired on the trip to the planet must be made up by a lead angle on the return trip.

The minimum trip time, of course, is attained by trajectories that permit t_w to be positive with $n = 0$. The conditions on trajectory angles and times to accomplish this reduction are, from eq. (30),

$$\frac{\theta_{T,1} + \theta_{T,2}}{t_1 + t_2} \gtrsim \dot{\theta}_E \text{ for } \dot{\theta}_E > \dot{\theta}_P \qquad (33)$$

and

$$\frac{\theta_{T,1} + \theta_{T,2}}{t_1 + t_2} \leq \dot{\theta}_E \text{ for } \dot{\theta}_E < \dot{\theta}_P . \qquad (34)$$

Relations (33) and (34) state that the mean angular velocity of the outward and inward trajectories must be greater than that of the Earth if the destination planet is farther from the sun, and must be less than that of the Earth if the planet is closer to the sun.

Basic Data on Solar System and Minimum-Energy Trajectories

Table I presents miscellaneous data on the solar system. Included are masses, gravitational constants, angular velocities, and distances needed for the interplanetary trajectory computations, as well as other information. The data were compiled from [3] to [6].

For minimum-energy orbits, the transit time is half the period of the ellipse having major axis a equal to the sum of the radii of the origin and the destination:

$$t_{T,\,m.e.} = \frac{\pi}{2}\sqrt{\frac{(r_E + r_P)^3}{2\,\mu}} = 64.55\left(1 + \frac{r_P}{r_E}\right)^{3/2}, \text{ days} \tag{35}$$

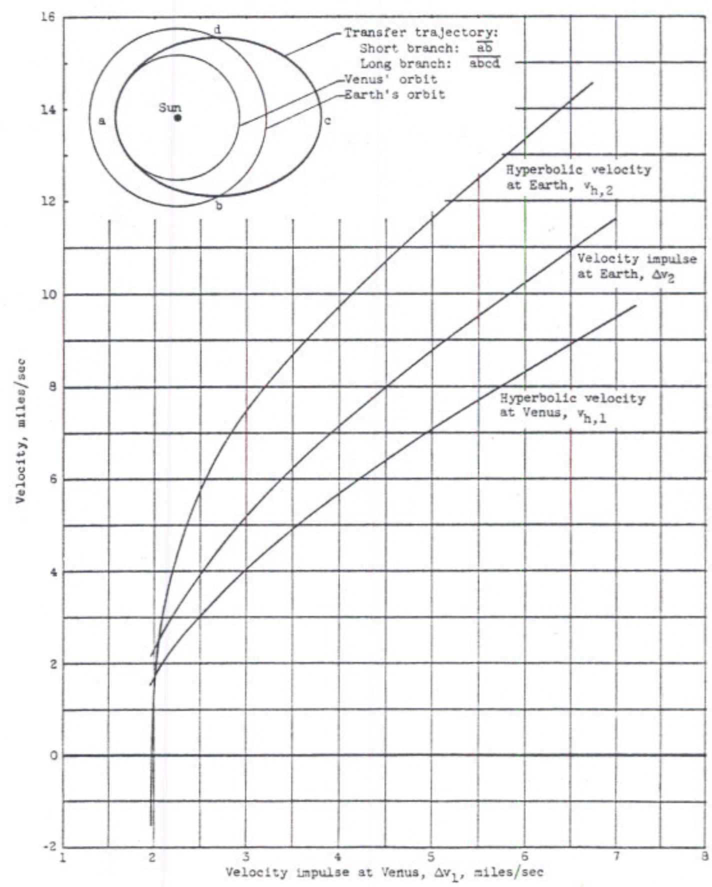

Fig. 10. c) Trajectory tangent at Venus' orbit: velocities

where r_E and r_P are, respectively, the orbital radius of the Earth and of the destination planet.

The hyperbolic velocity at the Earth's orbit required to reach the destination orbit is obtained as follows. From eq. (10),

$$V_0{}^2 = \left(\frac{v_0}{v_{c,E}}\right)^2 = \frac{2\,\varrho_m}{1 + \varrho_m} \tag{36}$$

where $\varrho_m = r_P/r_E$. Consequently, from eq. (25),

$$v_{h,E} = v_0 - v_{c,E} = v_{c,E}\left(\sqrt{\frac{2\,\varrho_m}{1 + \varrho_m}} - 1\right). \tag{37}$$

The hyperbolic velocity at the destination orbit is obtained from eqs. (11), (36), and (25). Thus, the heliocentric velocity at the destination orbit is

$$V^2 = \frac{v^2}{v_{c,E}^2} = V_0^2 - 2\left(1 - \frac{1}{\varrho_m}\right) = \frac{2\,\varrho_m}{1+\varrho_m} - 2\left(1 - \frac{1}{\varrho_m}\right) = \frac{2}{\varrho_m\,(1+\varrho_m)} \,. \qquad (38)$$

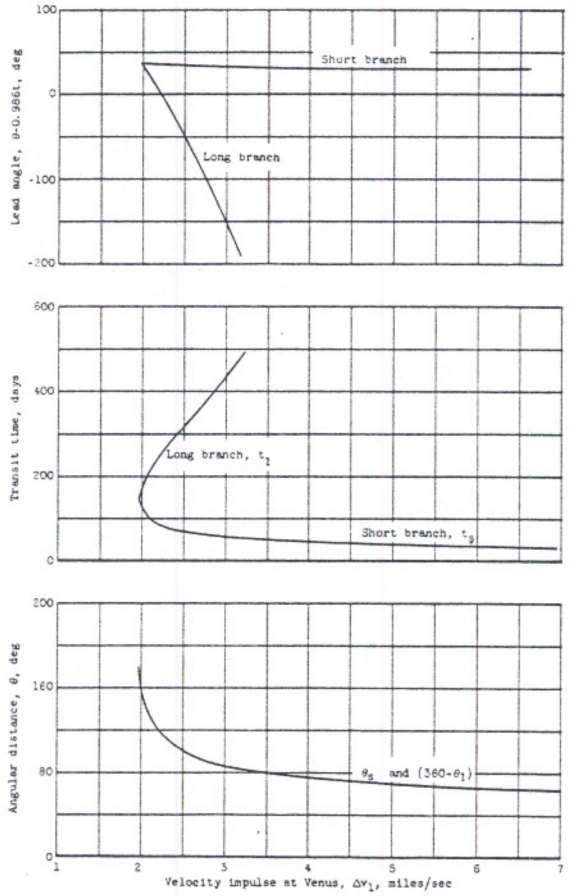

Fig. 10. *d*) Trajectory tangent at Venus' orbit: angular distance and time

Consequently, the hyperbolic velocity at the destination orbit is

$$v_{h,P} = v - v_{c,P} = v_{c,E}\sqrt{\frac{2}{\varrho_m\,(1+\varrho_m)}} - v_{c,P} \,. \qquad (39)$$

The velocity increments required to achieve these minimum-energy orbits are then obtained by substituting these hyperbolic velocities into eq. (27).

Table II presents information on least-energy interplanetary trajectories, including transit times, waiting times, and velocity impulses required to leave and enter circular satellite orbits at radii equal to 1.1 times the planetary radii. The data of Table II are useful for reference and for comparison with excess-energy values.

Table II. *Data Concerning Minimum-Energy Orbits from Earth*
All velocities in miles/sec.

Destination Body	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto	Moon
Hyperbolic velocity at Earth's orbit to attain orbit of destination	—4.66	—1.555	0	1.85	5.46	6.38	7.02	7.26	7.36	—
Hyperbolic velocity at destination orbit upon arrival from Earth	5.94	1.70	0	—1.65	—3.51	—3.37	—2.92	—2.51	—2.28	—0.514
Circular velocity at radius equal to 1.1 times surface radius	1.78	4.29	4.69	2.13	25.1	15.1	9.22	10.6	6.66	1.0
Gravity at 1.1 times surface radius, ft/sec²	10.2	23.3	26.6	10.5	69.8	30.5	27.7	29.8	11.8	4.4
Initial velocity impulse to attain destination orbit from satellite orbit around Earth at $r = 1.1\ R_0$, Δv_1	3.42	2.13	—	2.19	3.90	4.50	5.0	5.14	5.20	1.87
Velocity impulse at destination to establish satellite orbit at $r = 1.1\ R_0$, Δv_2	4.67	2.01	—	1.30	10.6	6.5	4.1	4.6	3.0	0.505
Total velocity impulses for round trip $2\,(\Delta v_1 + \Delta v_2)$	16.18	8.28	—	6.98	29.0	22.0	18.2	19.5	16.4	4.75
Transit time from Earth to destination, days	106.0	146.0	—	259.0	1000.0	2200.0	5850.0	11200	16600	5.0
Wait time at destination, days	69.6	468.0	—	455.0	208.0	315.0	246.0	224.0	204.0	0
Total trip time, days	281.6	760.0	—	973.0	2208.0	4755.0	12046	22624	33604	10.0

Trip Times as Function of Velocity Increments

One-Way Transit Times and Angular Distances

The transit times and angular trajectory distances between the Earth's orbit and the orbits of Venus and Mars have been calculated as a function of

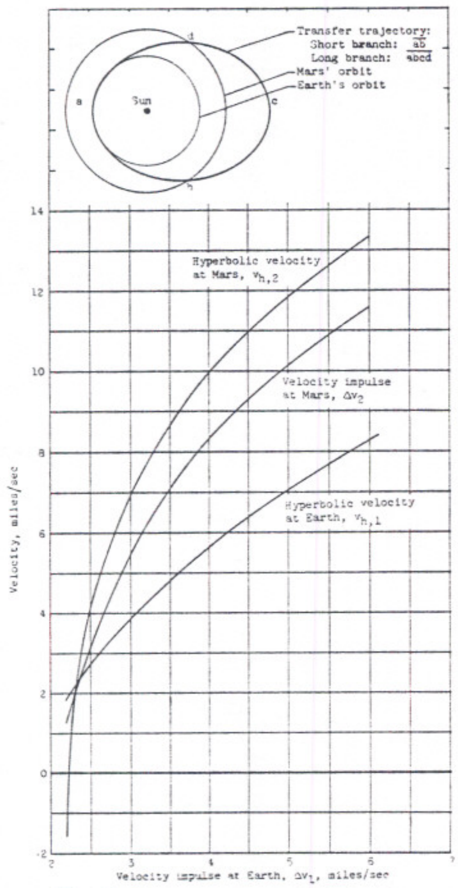

Fig. 11. Excess-energy trajectories between Earth and Mars, starting and ending in circular orbits at $r = 1.1\ R_{planet}$. *a*) Trajectory tangent at Earth's orbit: velocities

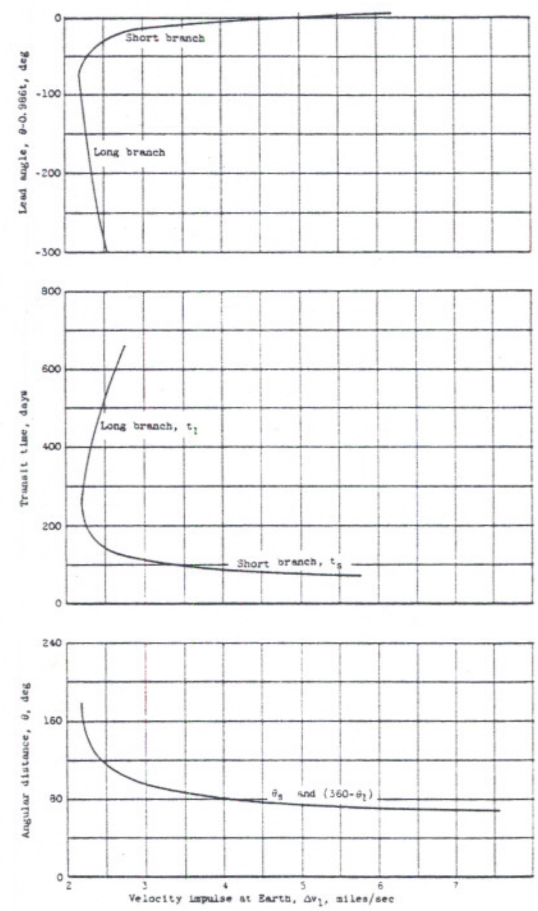

Fig. 11. *b*) Trajectory tangent at Earth's orbit: angular distance and time

the velocity increments applied in circular orbits with radius equal to 1.1 times the planetary radii. Results are shown for four families of trajectories: (1) trajectories between Venus and Earth tangent at Earth's orbit [Figs. 10 (a) and (b)], (2) trajectories between Venus and Earth tangent at Venus' orbit [Figs. 10 (c) and (d)], (3) trajectories between Earth and Mars tangent at Earth's orbit [Figs. 11 (a) and (b)], and (4) trajectories between Earth and Mars tangent at Mars' orbit [Figs. 11 (c) and (d)]. For each case, the first figure shows the hyperbolic velocities at the two orbits and the velocity increment required at the second planet as functions of the velocity increment applied at the planet whose orbit is tangent to the trajectory. The second figure shows the angular distance of the trajectory, the transit time, and the lead angle for both the long and the short branch of the transfer trajectory. (The hyperbolic velocities and the velocity increments are, of course, the same for the long and the short branches.)

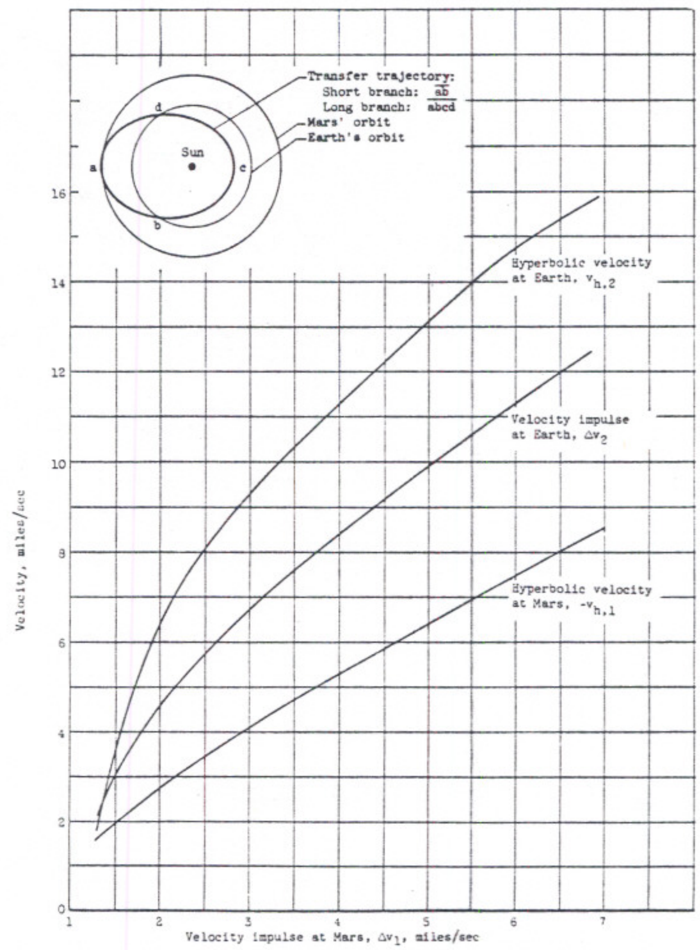

Fig. 11. *c*) Trajectory tangent at Mars' orbit: velocities

Total Trip Time as Function of Total Velocity Increments

With the basic one-way transit data of Figs. 10 and 11, the two-way transit time, the waiting time, and the total trip time can be found as functions of the sum of the four velocity increments required for the round trip. These quantities can be found for a variety of combinations of outward and inward paths. Two families of combinations have been selected for particular discussion, because both have two of the four hyperbolic velocities tangent to an orbit. Such combinations are likely to produce time reductions with minimum excess energy, because the Δv rises quite rapidly as the angle with which the trajectory crosses an orbit increases. However, no proof exists that the families selected are optimum.

Outward and inward trip along same ellipse.—The first family of combinations consists of those that follow one or the other branch of the same ellipse on the return trip as was followed on the trip to the planet. In this case, the two tangent velocities occur at the same planet. For this family, the waiting times at the

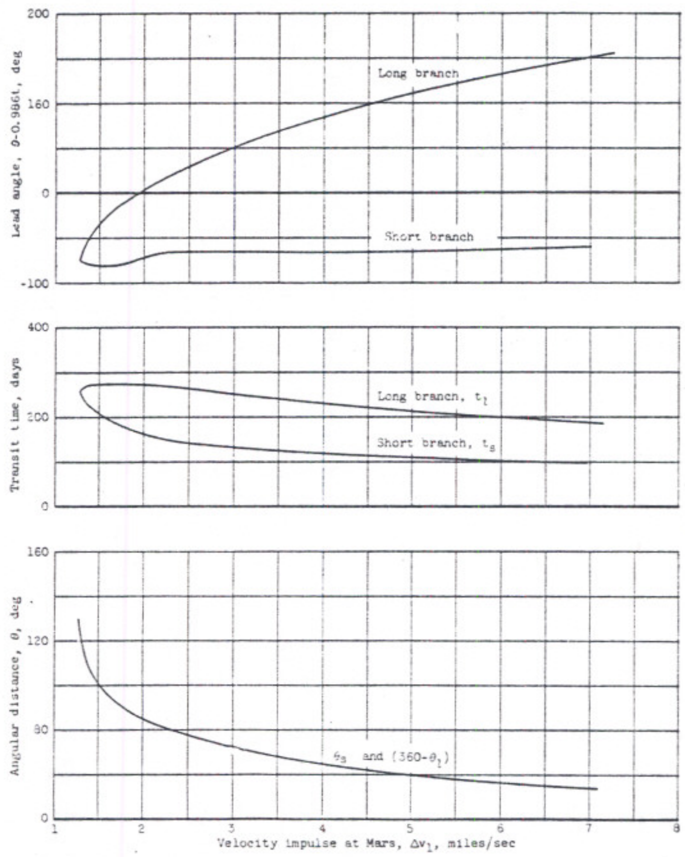

Fig. 11. *d*) Trajectory tangent at Mars' orbit: angular distance and time

destination planet cannot be arbitrarily specified, because the angular distance and the time required for the return trip have one of two values, depending on whether the long or the short branch of the ellipse is followed. For each ellipse, there are, therefore, three combinations of to and fro trips: (1) Trip to planet and return along short branch of ellipse (denoted by s—s); (2) trip to planet along short branch and return trip along long branch, or vice versa (denoted by s—l); (3) trip to planet and return along long branch (denoted by l—l).

Shown in Figs. 12 and 13 are the two-way transit times, the waiting times, and the total round-trip times as functions of the sum of the four velocity increments. In Fig. 12 (a), results are shown for the Earth—Venus trip along ellipses tangent to the orbit of Venus. Fig. 12 (b) shows the results for the Earth—Venus trip along ellipses tangent to the Earth's orbit. Fig. 13 (a) and (b) show similar results for the Earth—Mars trip.

The first substantial reduction in total trip time for the Venus trip occurs for a total Δv of 10.6 miles per second [Fig. 12 (a)]. The trip time is reduced from the least-energy value of 760 days to 520 days along a trajectory that follows the long branch of the ellipse for both trips. The next reduction in trip times

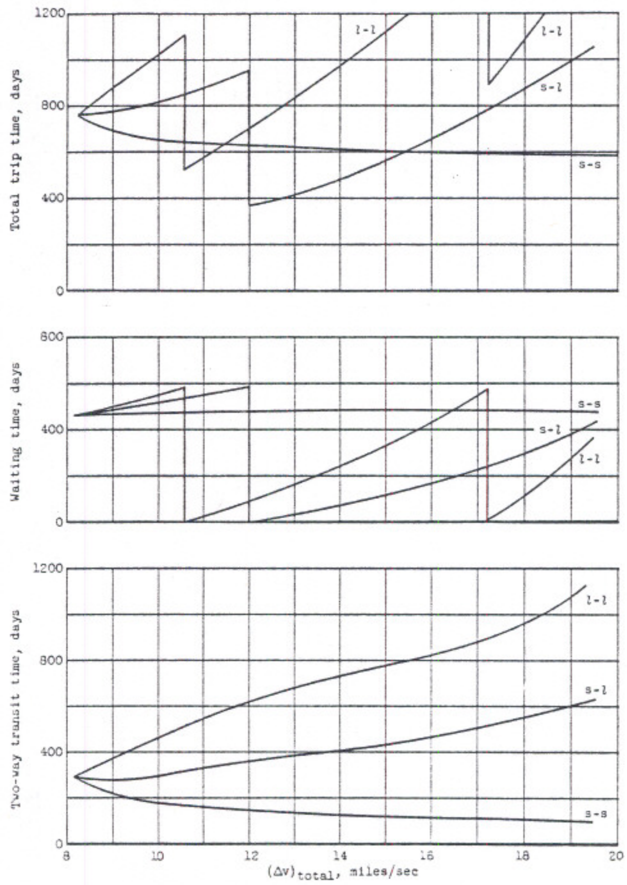

Fig. 12. Transit times, waiting times, and total round-trip times for excess-energy Venus trips along single ellipse. *a)* Case I: Trajectory tangent at Venus' orbit

occurs for $(\Delta v)_{total} = 12$ miles per second. This trajectory is an ellipse having the same period as the Earth and therefore yields a total trip time of 365 days with zero waiting time. One trip is along the short branch and the other along the long branch of the ellipse. An even greater reduction in total time occurs for a total Δv of 13.6 miles per second with short-short combination [Fig. 12 (b)]. The total time is reduced to 180 days, but very substantial increases are required to provide waiting times sufficient for useful exploration.

For the Mars trip, much larger total velocity increments are required for substantial reductions in trip times. The first breakthrough occurs with a total Δv of 12.4 miles per second [Fig. 13 (b)], for which the total trip time is reduced from the least-energy value of 973 days to 540 days. This first reduction, again, occurs for a long-long combination. The next substantial reduction in time occurs with $(\Delta v)_{total} = 26.2$ miles per second. The total time, with zero wait time at Mars, is 365 days, which again is an l—s trip along an ellipse having a period equal to that of the Earth. This ellipse has its perigee about 40×10^6 miles from the sun. A further reduction in total time to 160 days is achieved

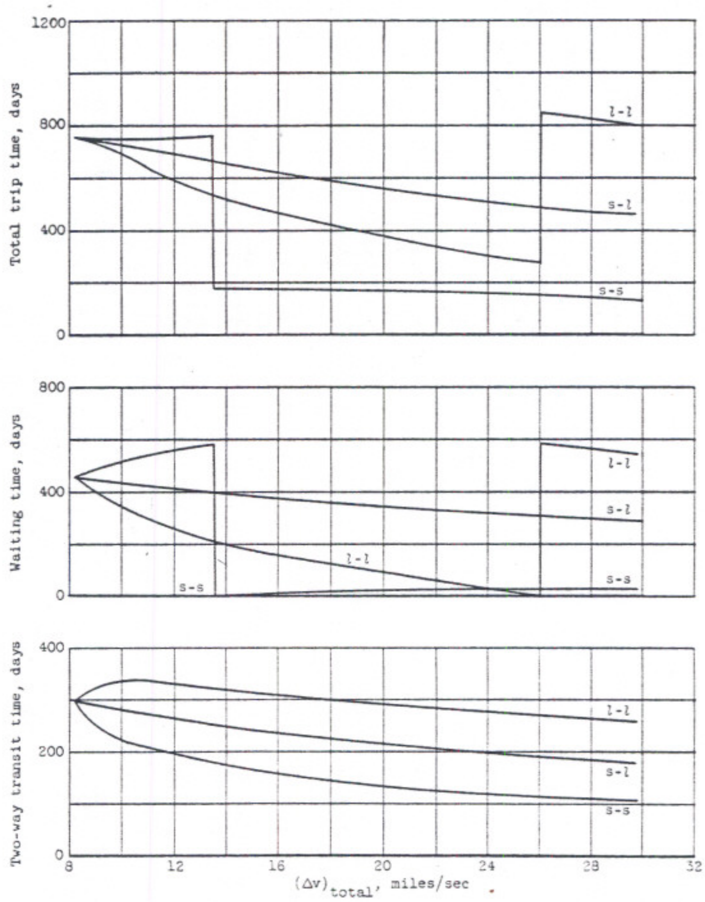

Fig. 12. *b*) Case II: Trajectory tangent at Earth's orbit

along an s—s trajectory with total $\Delta v = 29$ miles per second [Fig. 13 (a)]; however, sizable increases in Δv are required to provide adequate exploration time.

Trips along elliptic arcs tangent at Earth's orbit and at destination orbit.— The second family of trajectories having two tangent hyperbolic velocities consists of combinations for which one trajectory is tangent to the Earth's orbit and the other is tangent to the orbit of the destination planet. For this case, for any outward trip, the waiting time can be specified, and the lead angles of the return trajectories can be examined to determine which, if any, satisfies eq. (32). Results are shown in Fig. 14 (a) for the Earth—Venus trip and in Fig. 14 (b) for the Mars trip, both for waiting times of 0 and 100 days.

For both trips, no solutions were found using short branches of orbits tangent to Mars or Venus. This can be understood by examining Figs. 10 (b) and (d) and 11 (b) and (d). For the short branch of the ellipses tangent to Venus' orbit [Fig. 10 (d)], the lead angle is positive and greater than 30° for the Δv range covered. Neither the short nor the long branch of the ellipses tangent to the Earth's orbit [Fig. 10 (b)] produces lag angles of as much as 30° to balance the

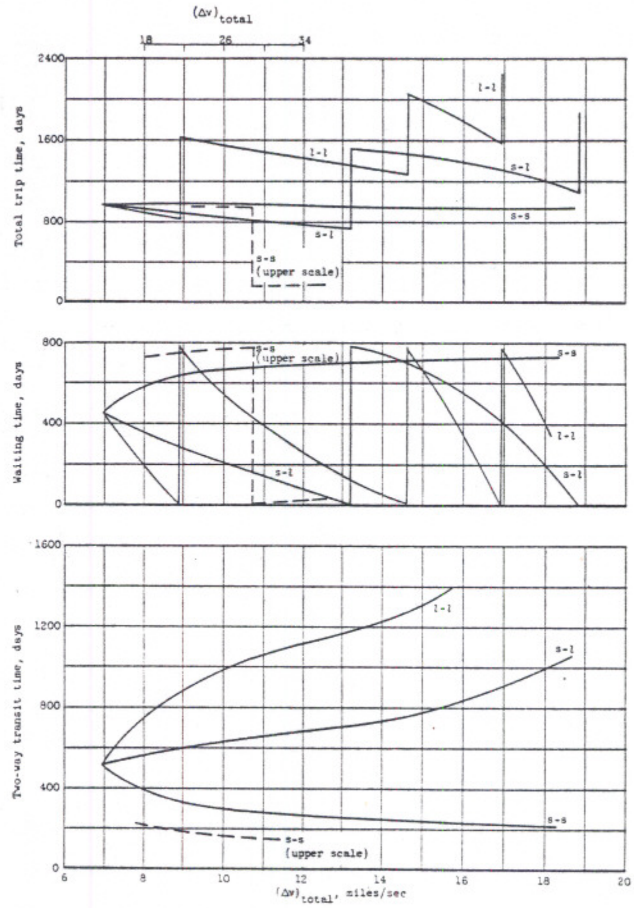

Fig. 13. Transit times, waiting times, and total trip times for excess-energy Mars trips along single ellipse. *a*) Case I: Trajectory tangent at Earth's orbit

30° lead angle. The long branch of the family of ellipses tangent to Venus, however, produces lead angles up to 30° and lag angles up to 200° or higher [Fig. 10 (d)]. These can balance all the lead angles for both the long and short branches of the ellipses tangent to the Earth's orbit. Similar considerations, with change of sign, apply for the short branch of ellipses tangent to Mars' orbit.

The solutions of Fig. 14 indicate that the l—l trips produce longer trip times for the same $(\Delta v)_{total}$ than the s—l trips and are therefore not of interest. The s—l trips, however, produce reductions in trip time with lower Δv than those obtained with the first family of combinations. (The best reductions for this family are indicated in Fig. 14 for comparison.) For zero waiting time, a trip-time reduction to 400 days is obtained with a total Δv of 9.7 miles per second for the Venus trip and with a total Δv of 14.9 miles per second for the Mars

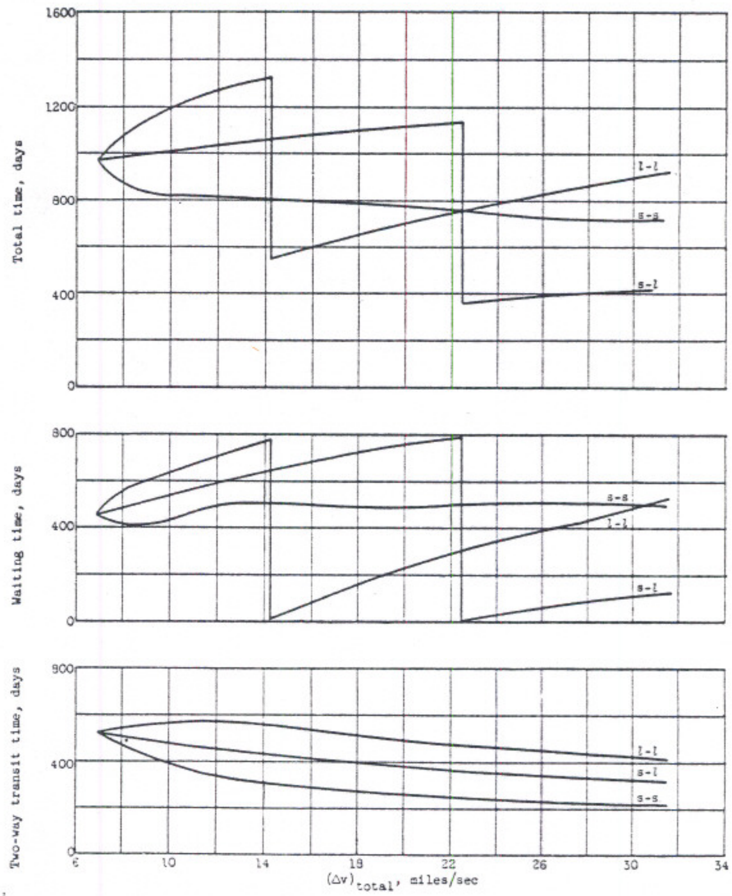

Fig. 13. *b*) Case II: Trajectory tangent at Mars' orbit

trip. For 100-day waiting time, the required Δv rises to 10.8 miles per second for the Venus trip and to 20 miles per second for the Mars trip.

Concluding Remarks

The combinations of excess-energy trajectories studies herein were limited to those attainable with a total of four velocity increments for the round trip, and with two of the four hyperbolic velocities tangent to the planetary orbits. These families appeared to offer the best possibility for reducing total trip time with the least excess energy. It is possible, however, that other trajectories can be found that require less energy for a given total trip time. Such families might include trajectories tangent at neither apsis, or trajectories with more than four velocity increments.

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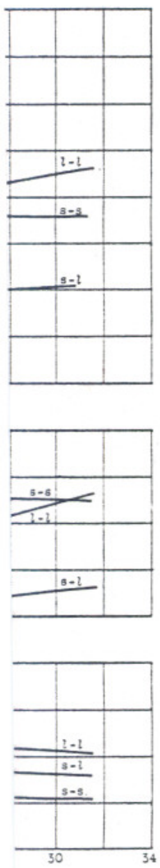

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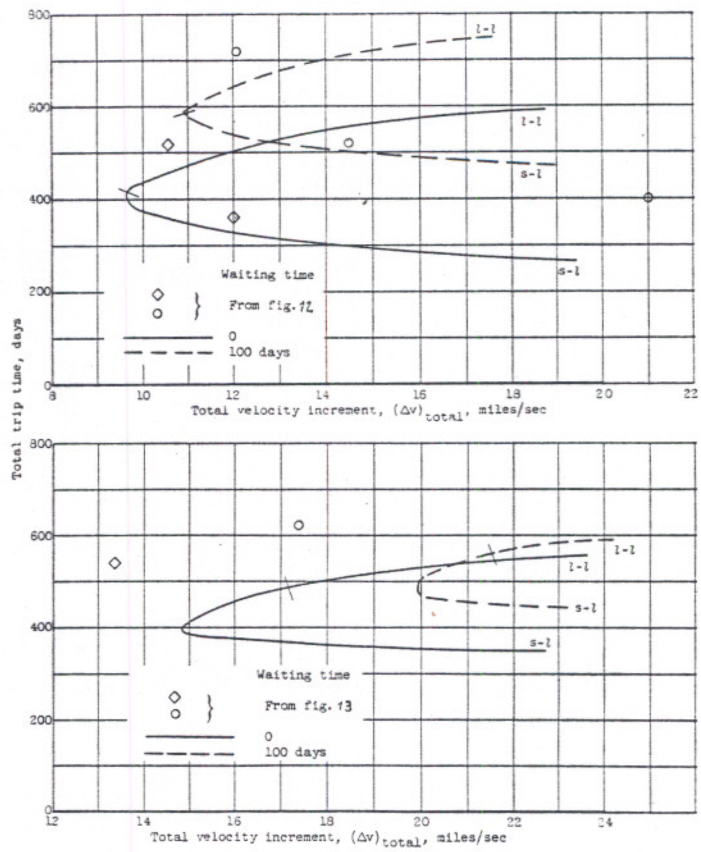

Fig. 14. Trip time for excess-energy trajectories along elliptic arcs tangent at Earth's orbit and at destination orbit. *a*) Earth—Venus round trip, satellite-to-satellite; *b*) Earth—Mars trip, satellite-to-satellite

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Addendum

After completion of this paper, the author found that another paper on round-trip interplanetary journeys had recently been presented[1]. In this paper are included results for a number of Mars trajectories, including most of those contained herein. However, no results were reported for the Venus trip.

[1] K. A. EHRICKE, M. D. WHITLOCK, R. L. CHAPMAN and C. H. PURDY, Calculations on a Manned Nuclear-Propelled Space Vehicle. Presented at the 12th Annual Meeting of the American Rocket Society, Dec. 2—5, 1957.